PHYS204 SUMMARY

by Augustin Winther ©

Version 1

Definition	Theory

Misc.

Date: 2025-05-16

Contact info at:
winther.io

SINGLE PARTICLE

Linear mechanics

$\mathbf{v} = \frac{d\mathbf{r}}{dt}$	$\mathbf{p} = m\mathbf{v}$	$\mathbf{a} = \frac{d^2\mathbf{r}}{dt}$
$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = m\mathbf{a}$		

$$\Sigma\mathbf{F} = 0 \Rightarrow \frac{d\mathbf{p}}{dt} = 0 \Rightarrow \mathbf{p} \text{ is conserved}$$

Angular mechanics

$\mathbf{L} = \mathbf{r} \times \mathbf{p}$	$\mathbf{N} = \mathbf{r} \times \mathbf{F} = \frac{d\mathbf{L}}{dt}$

$$\Sigma\mathbf{N} = 0 \Rightarrow \frac{d\mathbf{L}}{dt} = 0 \Rightarrow \mathbf{L} \text{ is conserved}$$

Work

From point 1 to point 2

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}$$

Work for const. mass

$$W_{12} = T_2 - T_1$$

where T is **Kinetic Energy**:

$$T = 1/2\, mv^2$$

Conservative force

If the path from 1 to 2 does not matter, then the force is conservative.

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0$$

Potential energy

If force is conservative, then $\mathbf{F}$ can be expressed as

$$\mathbf{F} = -\nabla V(\mathbf{r})$$

Differential work can further be expressed as

$$\mathbf{F} \cdot d\mathbf{s} = dV$$

Which implies

$$W_{12} = V_1 - V_2$$

where V is called the potential energy.

If forces acting on a particle are conservative, then **total energy of the particle**, $T + V$, is conserved.

SYSTEM OF PARTICLES

Newtons 2nd law

$$M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}$$

$\mathbf{R}$ is the center of mass, and M the total mass:

$$\mathbf{R} = \frac{\Sigma\, m_i \mathbf{r}_i}{M} = \frac{\Sigma\, m_i \mathbf{r}_i}{\Sigma m_i}$$

$\mathbf{F}^{(e)}$ is the external force acting on the system.

Linear momentum

$$\mathbf{P} = \sum m_i \frac{d\mathbf{r}_i}{dt} = M\frac{d\mathbf{R}}{dt}$$

$$\mathbf{F}^{(e)} = 0 \Rightarrow \frac{d\mathbf{P}}{dt} = 0 \Rightarrow \mathbf{P} \text{ is conserved}$$

Angular momentum

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \mathbf{R} \times \mathbf{P} + \sum_i \mathbf{r}_i{}' \times \mathbf{p}_i{}'$$

Where $\mathbf{r}'$ and $\mathbf{p}'$ are relative to the center of mass. The external **torque** is equal to

$$\mathbf{N}^{(e)} = \frac{d\mathbf{L}}{dt}$$

$$\mathbf{N}^{(e)} = 0 \Rightarrow \frac{d\mathbf{L}}{dt} = 0 \Rightarrow \mathbf{L} \text{ is conserved}$$

Kinetic Energy

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i'^2$$

where v is the velocity of center of mass, and v_i' is velocity of particle i relative to the center of mass.

Potential Energy

$$V = \sum_i V_i + \frac{1}{2}\sum_{\substack{i,j \\ i \neq j}} V_{ij}$$

where $V_{ij} = V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$ is a potential between two internal particles, where $\nabla V_{ij} = (\mathbf{r}_i - \mathbf{r}_j)f$, where f is some scalar function,

CONSTRAINTS

Holonomic

If the constraints can be expressed as

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, t) = 0$$

Example (rigid body): $(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0$

Non-holonomic

Particle on sphere for example:

$$r^2 - a^2 \geq 0$$

An inequality, not equality.

Generalized coordinates

System of N particles have $3N$ degrees of freedom. There exists k amount of holonomic constraint equations for the system. The system thus has $3N - k$ degrees of freedom. The generalized coordinates q_i represents these degrees of freedom, and every r_i can be expressed by them

$$\mathbf{r}_1 = \mathbf{r}_1(q_1, q_2, \ldots, q_{3N-k}, t)$$
$$\vdots$$
$$\mathbf{r}_N = \mathbf{r}_N(q_1, q_2, \ldots, q_{3N-k}, t)$$

Virtual displacement

Also called **infinitesimal variation**, noted as δr, shows how a system can hypothetically deviate very slightly form the actual path r without violating the constraints at a given time instant.

D'Alembert's principle

$$\sum_i \left(\mathbf{F}_i - \frac{d}{dt}\mathbf{p}\right) \cdot \delta\mathbf{r}_i = 0$$

where $\mathbf{F}_i$ is force on particle (excl. forces of constraints, and $\delta\mathbf{r}_i$ the virtual displacement.

LAGRANGIAN FORMULATION

Variational calculus techniques by example

Problem: find shortest path between two points in the plane.

The length of the curve is

$$I = \int_i^f ds = \int_i^f \sqrt{dx^2 + dy^2} = \int_i^f \sqrt{1 + y'^2}dx$$

where $y' = dy/dx$

We want to find a function $y = y(x)$ such that I is minimized. Let $\phi(y)$ be a **functional** of y.

$$\phi(y) = \sqrt{1 + y'^2} \qquad I = \int_i^f \phi(y)dx$$

There are infinitely many ways to draw a line between the points. All these paths can be expressed by

$$Y(x, \epsilon) = y(x) + \epsilon\eta(x)$$

where ϵ very small, and $\eta(i) = \eta(f) = 0$

The length of any possible curve is expressed by

$$I(\epsilon) = \int_i^f \phi(x, Y, Y')dx$$

We want to determine $y(x)$ such that I is **stationary** (i.e., $\delta I = 0$), which happens when

$$\left[\frac{d}{d\epsilon} I\right]_{\epsilon=0} = 0$$

We thus solve

$$\left[\frac{d}{d\epsilon}\int_i^f \phi(x, Y, Y')dx\right]_{\epsilon=0} = 0$$

which yields the **Euler-Lagrange equation**

$$\frac{d}{dx}\left(\frac{\partial\phi}{\partial y'}\right) - \frac{\partial\phi}{\partial y} = 0$$

Which has solution: $y = mx + b$

Hamilton's principle

The motion of the system from time t_1 to t_2 is such that the line integral of the Lagrangian (called **the action**):

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

is stationary. Meaning: $\delta S = 0$

Lagrangian and Lagrange's equations

$$\mathcal{L} = T - V$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) - \frac{\partial\mathcal{L}}{\partial q_j} = 0$$

which is also the **equations of motions**.

Note: $\mathcal{L}'(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}, t) + \frac{d}{dt}F(q, t)$ is also a Lagrangian that result in same equations of motion.

Monogenic

All forces *(except for constraint forces)* are derivable from the generalized scalar potential:

$$V(q_1, q_2, \dots, q_n;\ \dot{q}_1, \dot{q}_2, \dots \dot{q}_n;\ t)$$

More general Lagrange's equations

If not all the forces acting on the system are derivable from a potential *(such as with friction)* then Lagrange's equation can be written on the form

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) - \frac{\partial\mathcal{L}}{\partial q_j} = Q_j$$

Lagrange equations with dissipation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial\mathcal{F}}{\partial\dot{q}_j} = 0$$

Where $\mathcal{F}$ is the **Rayleigh's dissipation function**

$$\mathcal{F} = \frac{1}{2}\sum_i\left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right)$$

Cyclic/ignorable coordinates

If a coordinate q_j does not appear in the Lagrangian, then it is called cyclic or ignorable, and the Lagrange equation reduces to

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) = 0 \quad \text{or} \quad \frac{d}{dt}p_j = 0$$

Generalized velocity

By the chain rule:

$$\mathbf{v}_i \equiv \frac{d\mathbf{r}_i}{dt} = \sum_k \frac{\partial\mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial\mathbf{r}_i}{\partial t}$$

Generalized momentum

$$p_j = \frac{\partial\mathcal{L}}{\partial\dot{q}_j}$$

A.k.a *canonical momentum*

Generalized virtual displacement

$$\delta\mathbf{r}_i = \sum_j \frac{\partial\mathbf{r}_i}{\partial q_j}\delta q_j$$

No time variation δt is involved, since virtual displacement only considers displacement of coordinates.

Virtual work

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_j Q_j \delta q_j$$

Where Q_j are components of the **generalized force**

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial\mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial\mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Energy function

$$h(q_1, q_2, \dots, q_n;\ \dot{q}_1, \dot{q}_2, \dots \dot{q}_n;\ t) = \sum_j \dot{q}_j \frac{\partial\mathcal{L}}{\partial\dot{q}_j} - \mathcal{L}$$

If the transformation equations for generalized coordinates do not explicitly depend on time, AND the potential V does not depend on the generalized velocities, then, and only then, is $h = T + V = \text{"total energy"}$

Conservation of h

We have

$$\frac{dh}{dt} = -\frac{\partial\mathcal{L}}{\partial t}$$

Thus, if the Lagrangian does not explicitly dependent on time t (i.e., the variable t does not appear in $\mathcal{L}$), then we say that h is conserved.

CENTRAL FORCE

Two-Body → Reduced One-Body

Given a system consisting of two mass-points m_1 and m_2, where the only forces are due to an interaction potential $U(\mathbf{r})$, where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, we choose the six degrees of freedom to be the components of $\mathbf{R}$ and $\mathbf{r}$, where $\mathbf{R}$ is for the center of mass.

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(\mathbf{r}, \dot{\mathbf{r}}, \dots)$$

Where $M = m_1 + m_2$ is the total mass, and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the **reduced mass**.

Note that $\mathbf{R}$ is cyclic and will therefore not appear in the equations of motions. Meaning the center of mass is either still or moving uniformly. We can, therefore, instead just write

$$\mathcal{L}' = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(\mathbf{r}, \dot{\mathbf{r}}, \dots)$$

Reduced One-Body Equations of Motion

Assuming $U = V(r)$, where $r \equiv |\mathbf{r}|$, we are looking at a spherical symmetric problem

Rewriting $\mathcal{L}'$ with spherical coordinates, we have

$$\mathcal{L}' = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right)^2 - V(r)$$

This gives rise to these four equations

$$\mu r^2\dot{\theta} = l$$

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right)^2 + V(r)$$

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V - \frac{l^2}{2\mu r^2}\right)}}$$

$$\theta = l\int_0^t \frac{dt}{\mu r^2(t)} + \theta_0$$

Central force

A central force on an object is a force that is directed towards or away from a point called center of force.

Equation of orbit

Using the four equations of motion of the reduced one-body system, one can derive the integral

$$\theta = \theta' - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2\mu E}{l^2} - \frac{2\mu a}{l^2}u^{-n-1} - u^2}}$$

where $u = 1/r$, and $V(r) = ar^{n+1}$

If $V(r) = -k/r$ (called the **Kepler problem**) then the integral equates to the **elliptical orbit equation**:

$$\frac{1}{r} = \frac{\mu k}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos(\theta - \theta')\right)$$

This is the equation of a conic with one focus at the origin

$$\frac{1}{r} = C[1 + e\cos(\theta - \theta')]$$

Where $e = \sqrt{1 + \frac{2El^2}{\mu k^2}}$ is the **eccentricity**, and where we can see that θ' represents as one of the **turning angles** of the orbit.

The angular momentum and energy are **constants of orbit** (i.e., they decide the orbit)

Orbit properties

Eccentricity	Energy	Shape
$e > 1$	$E > 0$	Hyperbola
$e = 1$	$E = 0$	Parabola
$e < 1$	$E < 0$	Ellipse
$e = 0$	$E = -\mu k^2/(2l^2)$	Circle

Semi-major axis is given by

$$a = -\frac{k}{2E}$$

and one can write the elliptical orbit equation as

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \theta')}$$

Virial Theorem

$$\langle T\rangle = -\frac{1}{2}\sum_i \langle \mathbf{F}_i \cdot \mathbf{r}_i\rangle$$

Closed orbits

Orbits in which an object eventually retraces its own steps

Bertrand's theorem

Among central-force potentials with bound orbits, there are only two types of central-force potentials with the property that all bound orbits are also closed orbits:

$$V(r) = -\frac{k}{r} \quad \text{with force} \quad f(r) = -\frac{dV}{dr} = -\frac{k}{r^2}$$

$$V(r) = \frac{1}{2}kr^2 \quad \text{with force} \quad f(r) = -\frac{dV}{dr} = -kr$$

Orbit equation with time

If $V(r) = -k/r$, one can combine the elliptical orbit equation with the equation for time in the reduced one body set of equation to show that

$$t = \frac{l^3}{\mu k^2}\int_{\theta_0}^{\theta} \frac{d\theta}{[1 + e\cos(\theta - \theta')]^2}$$

If $e = 1$, we get

$$t = \frac{l^3}{2mk^2}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right)$$

If $e < 1$, we get

$$\tau = 2\pi a^{3/2}\sqrt{m/k}$$

where τ is the period of orbit. For the planets around the Sun, this becomes **Kepler's 3rd law**

$$\tau = \frac{2\pi a^{3/2}}{\sqrt{G(m_P + m_S)}} \approx \frac{2\pi a^{3/2}}{\sqrt{Gm_S}}$$

Laplace-Runge-Lenz vector

For the Kepler problem, we have a conserved vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu k\frac{\mathbf{r}}{r}$$

$$A^2 = \mu^2k^2 + 2\mu El^2$$

SCATTERING

Intensity
Number of particles crossing a unit area normal to the particle beam in per unit time. Often denoted by I

Solid angle
$$d\Omega = 2\pi \sin(\theta) d\theta$$
Where θ is the angle between the scattered and incident directions, known as the **scattering angle**.

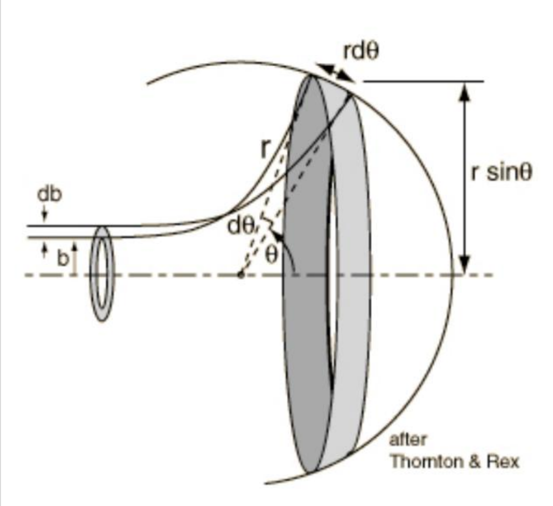
Cross section
$$\sigma(\boldsymbol{\Omega})d\Omega = \frac{\#\text{ particles scattered into } d\Omega \text{ per unit time}}{\text{incident intensity}}$$
Where $d\Omega$ is the **solid angle** in $\boldsymbol{\Omega}$ direction, and $\sigma(\boldsymbol{\Omega})$ is the **differential scattering cross section**

Angular momentum
Amount of scattering of particle is determined by its energy and angular momentum
$$l = mvb = b\sqrt{2mE}$$
where b is the **impact parameter**, where
$$b = b(\theta, E)$$

Differential cross section
$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Total cross section
$$\sigma_T = \int_{4\pi} \sigma(\boldsymbol{\Omega}) d\Omega$$
$$= 2\pi \int_{4\pi} \sigma(\theta) \sin\theta d\theta$$

Orbit equation Coulomb field
$$\frac{1}{r} = \frac{\mu\, Ze\, Z'e}{l^2}(\epsilon \cos\theta - 1)$$
Which is a hyperbolic orbit representing the scattering from a scatter scenter with potential
$$V(r) = -\frac{k}{r} = -\frac{Ze\, Z'e}{r}$$
Where Ze is the fixed charged responsible for scattering the incident particle with $Z'e$ charge



Source: https://www.tcm.phy.cam.ac.uk/~bds10/aqp/lec20-21.pdf

Cross section Coulomb field
Impact parameter function becomes
$$b = \frac{ZeZ'e}{2E}\cot\left(\frac{\theta}{2}\right)$$
Differential cross section becomes
$$\sigma(\theta) = \frac{1}{4}\left(\frac{ZeZ'e}{2E}\right)^2 \csc^4\left(\frac{\theta}{2}\right)$$

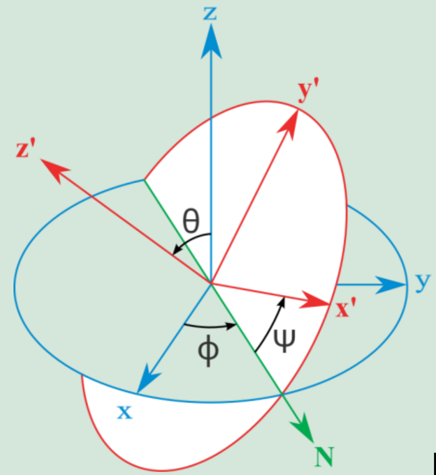
Rotation matrices
First rotation: $\mathbf{D} =$
$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Second rotation: $\mathbf{C} =$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$
Third rotation: $\mathbf{B} =$
$$\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Full transformation
$$\mathbf{A} = \mathbf{BCD}$$

SUMMATION CONVENTION

Einstein summation convention
Whenever an index occurs two or more times in a term, it is implied that the terms are to be summed over all possible values of the index. Example
$$a_{ij}x_j = \sum_{j=1}^{3} a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$$

THE EULER ANGLES ϕ ψ θ

One coordinate system relative to another


[1]

The **body axes** (x', y', z') can be described relative to the **space axes** (x, y, z) by angles of rotation. Visualized here by subsequent rotations of a white disc which in the end represents the (x', y', z') system. *Note that they have the same origin.* $\mathbf{N}$ is the **line of nodes** which is perpendicular to both the z-axis and the z'-axis

Space to Body axes transformation
$$\mathbf{A} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

Body to Space axes transformation
$$\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$$
$$= \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{bmatrix}$$

ROTATIONS of RIGID BODIES

Eulers Theorem

The general displacement of a rigid body with one point fixed, is a rotation Φ about some axis $\mathbf{R}$.

This axis has to be the same in the body and space axes:

$$\mathbf{R}' = \mathbf{A}\mathbf{R} = \mathbf{R}$$

Finding axis of rotation

Set $\lambda = 1$ in the eigenvalue equation $(\mathbf{A} - \lambda\mathbf{1})\mathbf{R} = 0$ and solve for $\mathbf{R}$

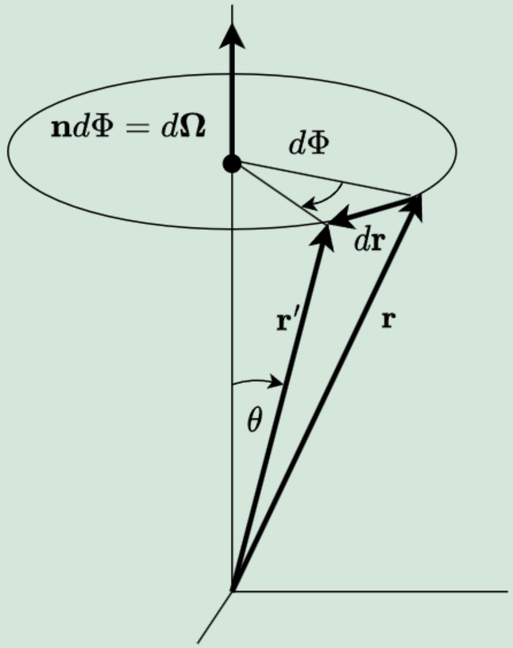
Finding angle of rotation

Solve the following for Φ

$$\cos\frac{\Phi}{2} = \cos\frac{\phi+\psi}{2}\cos\frac{\theta}{2}$$

Infinitesimal transformation of vector

$$d\mathbf{r} = \mathbf{r} \times d\mathbf{\Omega} \qquad \mathbf{r}' = \mathbf{r} + d\mathbf{r}$$



Where $\mathbf{n}d\Phi$ is along the axis of rotation, and $|d\mathbf{r}| = dr = r\sin\theta d\Phi$

Infinitesimal change in vector

Let $\mathbf{G}$ be a vector or pseud-vector in a mechanical problem *(e.g. position of point, total ang. mom. etc.)*

$$(d\mathbf{G})_{\text{space}} = (d\mathbf{G})_{\text{body}} + d\mathbf{\Omega} \times \mathbf{G}$$

$$\Downarrow$$

$$\left(\frac{d\mathbf{G}}{dt}\right)_{\text{space}} = \left(\frac{d\mathbf{G}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{G}$$

where $\boldsymbol{\omega}dt = d\mathbf{\Omega}$ e **instantaneous angular velocity** of the body

A very important operator is therefore

$$\boxed{\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \boldsymbol{\omega} \times}$$

Subscript s for "space". Subscript r for "rotating" i.e., the body

ω in Euler angles

$\boldsymbol{\omega}$ can be represented as three successive infinitesimal rotations with ang. velocities:

$\omega_\phi = \dot{\phi}$ along the z-axis.

$\omega_\theta = \dot{\theta}$ along the line of nodes.

$\omega_\psi = \dot{\psi}$ along the z'-axis.

Using the $\mathbf{B}, \mathbf{C}$, and $\mathbf{D}$ one can expressed these along any of the 6 desired axes.

$\boldsymbol{\omega}$ with respect to the <u>body axes</u> are

$$\boxed{\begin{aligned}\omega_{x'} &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_{y'} &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_{z'} &= \dot{\phi}\cos\theta + \dot{\psi}\end{aligned}}$$

Coriolis Effect

Consider distant nearly static stars as defining the spaces axes, and the rotating earth as the rotating system. $\boldsymbol{\omega}$ is the angular velocity of the earth.

$$\left(\frac{d\mathbf{r}}{dt}\right)_s = \left(\frac{d\mathbf{r}}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{r}$$

Can be written as

$$\mathbf{v}_s = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a}_s = \left(\frac{d\mathbf{v}_s}{dt}\right)_s = \left(\frac{d\mathbf{v}_s}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{v}_s$$

Which yields

$$\mathbf{a}_s = \mathbf{a}_r + 2(\boldsymbol{\omega} \times \mathbf{v}_r) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Newton's 2nd law states

$$\mathbf{F} = m\mathbf{a}_s$$

Which leads to

$$\mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}_r) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\mathbf{a}_r$$

This means, that to an observer in the rotating system (i.e., earth), it appears as if the particle is moving under the influence of an **effective force** equal to

$$\mathbf{F}_{\text{eff}} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}_r) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

The last term $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the **centrifugal force**.
The middle term $-2m(\boldsymbol{\omega} \times \mathbf{v}_r)$ is the **Coriolis effect**.

This does not only apply to earth, this applies to all rigid bodies rotating.

Chasles' Theorem

Any general displacement of a rigid body can be represented by a translation plus rotation.

The six degrees of freedom are often given as two sets:

1. Three Cartesian coordinates to describe translational motion
2. Three Euler angles to describe rotation.

If one point of the body is fixed, then this reduces to Eulers theorem.

RIGID BODY INERTIA

Splitting of energies

By Chasles' theorem, we can very often split the energies into two sets

$$T = T_{\text{tran}} + T_{\text{rot}} \qquad V = V_{\text{tran}} + V_{\text{rot}}$$

where $T_{\text{tran}} = \frac{1}{2}Mv^2$

with M being total mass, and v being velocity of the center of mass.

Total angular momentum

$$\mathbf{L} = m_i\left[\omega \mathbf{r}_i^2 - \mathbf{r}_i(\mathbf{r}_i \cdot \boldsymbol{\omega})\right]$$

Moment of inertia elements

The total angular momentum $\mathbf{L}$ can be expressed as

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

The diagonal I elements are **moment of inertia coefficients**, and the off-diagonal I elements are **products of inertia**.

Letting α and β represent components of $\boldsymbol{\omega}$ and $\mathbf{r}_i$, the I elements are given by

Rigid body of discrete particles:

$$\boxed{I_{\alpha\beta} = m_i\left(\delta_{\alpha\beta} r_i^2 - r_{i\alpha} r_{i\beta}\right)}$$

Rigid continuous body:

$$\boxed{I_{\alpha\beta} = \int_V \rho(\mathbf{r})\left(r^2 \delta_{\alpha\beta} - r_\alpha r_\beta\right) dV}$$

Where $\delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$ is the **Kronecker-Delta**, and

$\rho(\mathbf{r})$ is the density at $\mathbf{r}$. If uniform density, then $\rho = \text{const}$

Moment of inertia tensor $\overleftrightarrow{\mathbf{I}}$

$$\overleftrightarrow{\mathbf{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \text{ where } \mathbf{L} = \overleftrightarrow{\mathbf{I}}\boldsymbol{\omega}$$

Note that $I_{\alpha\beta} = I_{\beta\alpha}$ when $\alpha \neq \beta$

Moment of inertia about axis of rotation

$$I = \mathbf{n} \cdot \overleftrightarrow{\mathbf{I}} \cdot \mathbf{n} = m_i\left[r_i^2 - (\mathbf{r}_i \cdot \mathbf{n})^2\right]$$

where $\mathbf{n}$ is the unit vector defined by $\boldsymbol{\omega} = \omega\mathbf{n}$

Can also be expressed as:

$$I = m_i(\mathbf{r}_i \times \mathbf{n}) \cdot (\mathbf{r}_i \times \mathbf{n})$$

or

$$I = \frac{m_i}{\omega^2}(\boldsymbol{\omega} \times \mathbf{r}_i) \cdot (\boldsymbol{\omega} \times \mathbf{r}_i)$$

Rotational kinetic energy

$$T_{\text{rot}} = \frac{1}{2} m_i (\boldsymbol{\omega} \times \mathbf{r}_i)^2 = \frac{1}{2} I \omega^2 \quad \Rightarrow \quad I = \frac{2T_{\text{rot}}}{\omega^2}$$

Letting α and β represent components of $\boldsymbol{\omega}$ and $\mathbf{r}_i$ we can also write

$$T_{\text{rot}} = \frac{1}{2} I_{\alpha\beta}\omega_\alpha\omega_\beta$$

Parallel axis theorem

The moment of inertia about a given axis, depend only on the moment of inertial about a parallel axis through the center of mass.

Let I_{cm} be the moment of inertia about the axis through the center of mass. Let I_a be the moment of inertia about a axis a parallel to the axis through the center of mass. We then have

$$\boxed{I_a = I_{\text{cm}} + M(\mathbf{R} \times \mathbf{n})^2 = I_{\text{cm}} + MR^2 \sin^2\theta}$$

Where $\mathbf{R}$ is the center of mass position, and M is the total mass. The angle θ is the angle between $\mathbf{R}$ and $\mathbf{n}$, or equivlently the angle between $\mathbf{R}$ and $\boldsymbol{\omega}$.

Principal inertias

There exists a set of coordinates in which $\overleftrightarrow{\mathbf{I}}$ is diagonal with the three principal inertias

$$I_1 = I_{xx} \qquad I_2 = I_{yy} \qquad I_3 = I_{zz}$$

In these set of coordinates, the rot. kinetic energy is

$$T_{\text{rot}} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

Principle moment of inertia tensor

One can construct a transform. matrix $\mathbf{A}$ such that

$$\overleftrightarrow{\mathbf{I}}_D = \mathbf{A}\overleftrightarrow{\mathbf{I}}\mathbf{A} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

where $\mathbf{A}$ can be express in Eulers angles as shown earlier. The direction of the axes x', y' and z' defined by $\mathbf{A}$ are the **principal axes**.

More mathematically speaking

I_1, I_2, and I_3 are the eigenvalues of $\overleftrightarrow{\mathbf{I}}_D$, and the direction vectors of the axes x', y' and z' are the corresponding eigenvectors.

Finding principal inertias and axes

They are the roots of the secular cubic equation arising from

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = \mathbf{0}$$

For each of the roots above, one can solve

$$\overleftrightarrow{\mathbf{I}}_D = \mathbf{A}\overleftrightarrow{\mathbf{I}}\mathbf{A}$$

to obtain the direction of the principal axes.

Finding by inspection:

If the rigid body is a solid of revolution about some axis, where the origin of the body system is on the axis of symmetry, then the principal axes are the axis of symmetry and the two perpendicular axes located in the plane normal to the axis of symmetry.

The **axis of symmetry** is the axis you can rotate about without the body changing appearance.

Transforming inertial tensor

If the principal moment of inertia tensor $\overleftrightarrow{\mathbf{I}}_D$ is known, one can find the inertia tensor $\overleftrightarrow{\mathbf{I}}$ in any other set of axes through the center of mass by transformation

$$\overleftrightarrow{\mathbf{I}} = \mathbf{S}\overleftrightarrow{\mathbf{I}}_D\mathbf{S}^\intercal$$

Where $\mathbf{S}$ is the transformation matrix relating the principle set of axes and the new set of axes.

RIGID BODY EQUATIONS OF MOTION

Splitting of Lagrangian

Given we can write $T = T_{\text{tran}} + T_{\text{rot}}$ and $V = V_{\text{tran}} + V_{\text{rot}}$, we can also write the Lagrangian as

$$\mathcal{L}(q,\dot{q}) = \mathcal{L}_c(q_c,\dot{q}_c) + \mathcal{L}_b(q_b,\dot{q}_b)$$

Where $\mathcal{L}_c$ involves the generalized coordinates of the center of mass q_c, and $\mathcal{L}_b$ involves the generalized coordinates of the orientation of the body about the center of mass q_b

Eulers equations

Working in terms of the system defined by the principal axes. If given a rigid body with rotational motion about the center of mass, or a fixed point, one can use Newtons 2nd law

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{s}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{b}} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N}$$

to derive the following equations of motion

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

If there are no net forces or torques applied on the rigid body, then $N_1 = N_2 = N_3 = 0$, then the center of mass is either still of moving at constant speed.

Figure axis

The principal axis with highest moment of inertia.

Symmetric top

Rotating rigid body with two equal principal inertias.

Potential in gravitational field

In a uniform gravitational field, the potential of a body is the same as if the body was concentrated at the center of mass

Heavy symmetrical top with one point fixed

Let (x,y,z) be the body axes, and (x',y',z') the space axes.

The symmetry axis is designated as the z axis and is one of the principal axes.

The top is fixed at one point; thus, its configuration is completely specified by the Euler angles:

$\theta =$ The inclination of z axis relative to z'-axis
$\phi =$ The azimuth of the top about the z'-axis
$\psi =$ The rotating angle of the top about the z-axis

where the z-axis is the same as the vertical.

The tops characteristic of motion is given by

$\dot{\theta} =$ bobbing up and down of the z axis relative to the z'-axis.
$\dot{\phi} =$ precession/rotation of the z axis about the z'-axis.
$\dot{\psi} =$ spinning/rotation of the top about the z-axis

For many cases, such as this, we have $\dot{\psi} \gg \dot{\theta} \gg \dot{\phi}$

The top is symmetric, meaning $I_1 = I_2 \neq I_3$, and kinetic energy becomes

$$T = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2$$

Written in Euler angles is equal to

$$\boxed{T = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2}$$

Let $\mathbf{R}$ be the location of the center of mass, and M be the total mass of the top, the potential energy becomes

$$\boxed{V = -M\mathbf{R}\cdot\mathbf{g} = Mgl\cos\theta}$$

Where l is the distance from the $x'y'$-plane to the center of mass

---

The Lagrangian becomes

$$\boxed{\mathcal{L} = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgl\cos\theta}$$

And the system is conservative

$$E = T + V = \text{const.}$$

The **generalized momentum to a rotation angle** is the component of the total angular moment along the axis of rotation. Example, if ψ is about the z-axis in the body, then $p_\psi = L_z$;

Starting to solve the heavy symmetrical top with one point fixed

The torque of gravity $\mathbf{N} = \mathbf{R}\times\mathbf{g}$ is along the line of nodes, since $\mathbf{R}$ is along z' and $\mathbf{g}$ is along z, and the line of nodes is perpendicular to both z and z'. Thus, the torque along z and z' is zero, and the angular momentum along these axes must be constant since $d\mathbf{L}/dt = \mathbf{N}$

ψ is rotation around z, thus $p_\psi = L_3 = I_3\omega_3$. L_3 must be constant, and we can therefore write $p_\psi = I_1 a$ where a some constant.

ϕ is rotating around z' axis, and we can do the same for that one $p_\phi = I_1 b$ where b is const.

Using $p_\psi = I_3\omega_3 = I_1 a$ and $p_\phi = I_1 b$ we show:

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}, \quad \dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta\frac{b - a\cos\theta}{\sin^2\theta}$$

what remains is the function $\theta = \theta(t)$

The above equations can be put into E, and we can write

$$\alpha = \dot{\theta}^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta$$

where $\alpha = (2E - I_3\omega_3^2)/I_1$ and $\beta = 2Mgl/I_1$

Letting $u = \cos\theta$ this can be further reduced to $\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^3$

Finding a function $\theta = \theta(t)$ can then be done by solving the following integral (by computer)

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(1-u^2)(\alpha - \beta u) - (b - au)^3}}$$

When $\beta = 0$ we are dealing with a gyroscope, when $\beta > 0$ we are dealing with a spinning top. A bit of information can be gathered by analyzing the function under the root:

$$f(u) = \dot{u}^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2)$$

SPECIAL RELATIVITY

All frames discussed here are assumed to be **inertial frames**, i.e., Newtons 1st law holds.

Special theory of relativity postulates

1. The laws of physics are the same in all inertial frames.
2. The speed of light is the same in all inertial frames.

Event in spacetime

Something at a time t and at a location $\mathbf{r} = (x, y, z)$

It can be represented by the vector $(ct, \mathbf{r}) = (ct, x, y, z)$ in Minkowski spacetime

Spacetime interval

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)$$

The deltas represent differences between two events A and B. ($\Delta t = t_B - t_A$, $\Delta x = x_B - x_A$ etc.)

Infinitesimal spacetime interval

$$(ds)^2 = (c\,dt)^2 - (dx^2 + dy^2 + dz^2)$$

$(ds)^2 < 0 \Rightarrow$ **Spacelike** $\quad (ds)^2 = 0 \Rightarrow$ **Lightlike**

$(ds)^2 > 0 \Rightarrow$ **Timelike**

Invariant spacetime interval

$(ds)^2$ is the same in all inertial reference system as it describes a geometric quantity of Minkowski spacetime.

Proper time and laboratory time

Proper time of a body is the time measured by a clock at rest with respect to that body.

Laboratory time is the time measured by any other clock not at rest with respect to that body.

Time dilation

Consider a body at rest in frame S' with proper time τ. If the frame S' is moving with speed $\mathbf{v}$ relative to another frame S, the laboratory time t measured in S would be

$$t = \tau/\sqrt{1-(v/c)^2} \;\Rightarrow \tau < t$$

Homogeneous Lorentz boost along one axis

Given two reference frames S and S' with parallel axes, and whose origin are the same at $t = t' = 0$. If S' travels at a speed v relative to S along the common x-axis, then we have

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

$\gamma = 1/\sqrt{1-\beta^2}$ is the **Lorentz factor** and $\beta = v/c$

General homogeneous Lorentz boost

Requires the axes of S and S' to be parallel.

Four-vector form:

$$ct' = \gamma(ct - \boldsymbol{\beta}\cdot\mathbf{r})$$

$$\mathbf{r}' = \mathbf{r} + \frac{(\boldsymbol{\beta}\cdot\mathbf{r})\boldsymbol{\beta}(\gamma-1)}{\beta^2} - \boldsymbol{\beta}\gamma ct$$

where $\boldsymbol{\beta} = \frac{\mathbf{v}}{c} = \frac{1}{c}(v_x, v_y, v_z) = (\beta_x, \beta_y, \beta_z)$

Matrix form:

$(ct', x', y', z') = \mathbf{x}' = \mathbf{L}\mathbf{x} = \mathbf{L}(ct, x, y, z)$ were

$$\mathbf{L} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & G\beta_x^2+1 & G\beta_y\beta_x & G\beta_z\beta_x \\ -\gamma\beta_y & G\beta_x\beta_y & G\beta_y^2+1 & G\beta_z\beta_x \\ -\gamma\beta_z & G\beta_x\beta_z & G\beta_y\beta_z & G\beta_z^2+1 \end{bmatrix}$$

with $G = \gamma^2/(1+\gamma)$

To "flip" the boost, change the sign of the β values.

Velocity addition via Lorentz boosts (one axis)

Given three frames S_1, S_2 and S_3 all with parallel x-axis. Let S_2 move with speed v relative to S_1, and let S_3 move with speed v' relative to S_2. The Lorentz boost from S_1 to S_3 is given by $\mathbf{L}_{1\to3} = \mathbf{L}_{2\to3}\mathbf{L}_{1\to2}$ which equal

$$\mathbf{L}_{1\to3} = \begin{bmatrix} \gamma\gamma'(1+\beta\beta') & -\gamma\gamma'(\beta+\beta') & 0 & 0 \\ -\gamma\gamma'(\beta+\beta') & \gamma\gamma'(1+\beta\beta') & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

General homogeneous Lorentz transformation

Given by a Lorentz boost $\mathbf{L}_0$, followed by a rotation $\mathbf{R}$: $\quad \mathbf{L} = \mathbf{R}\mathbf{L}_0$

Pure Lorentz boost matrices are symmetric. If a Lorentz transform. not symmetric, it has a rotation.

Thomas precession rotation

Consider frames S_1, S_2, and S_3. S_2 is moving with velocity $\boldsymbol{\beta}$ relative to S_1, and S_3 is moving with velocity $\boldsymbol{\beta}''$ relative to S_2. Additionally, S_3 is moving with velocity $\boldsymbol{\beta}''$ relative to S_1

Without loss of generality, we arrange the S_1 axes such that $\boldsymbol{\beta}$ is along the x-axis of S_1, and $\boldsymbol{\beta}'$ is in the $x'y'$-plane of S_2.

Both $\mathbf{L}_{1\to2}$ and $\mathbf{L}_{2\to3}$ are symmetric, but the total transformation $\mathbf{L}_{1\to3} = \mathbf{L}_{2\to3}\mathbf{L}_{1\to2}$ is not, and must therefore correspond to a rotation $\mathbf{R}$ and a boost. Notably, all off-diagonal values related to z in $\mathbf{L}_{1\to3}$ are zero, implying a rotation about z-axis.

Assuming $\boldsymbol{\beta}'$ is small compared to $\boldsymbol{\beta}$, and small compared to c, we can approximate

$$\mathbf{L}_{1\to3} \approx \begin{bmatrix} \gamma'' & -\gamma''\beta_x'' & -\gamma''\beta_y'' & 0 \\ -\gamma''\beta_x'' & \gamma'' & 0 & 0 \\ -\gamma''\beta_y'' & -\gamma''\beta_x''\beta_y'' & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And the rotation is

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (\gamma-1)\beta_y''/\beta & 0 \\ 0 & -(\gamma-1)\beta_y''/\beta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Implying that S_3 is rotated with respect to S_1 around the z-axis by an infinitesimal angle

$$\boxed{\Delta\Omega = (\gamma-1)\beta_y''/\beta}$$

Consider an **accelerating particle** in S_1. We imagine infinitely many inertial frames moving in S_1, each representing the instantaneous velocity $\mathbf{v}$ of the particle at a given time. Let S_2 and S_3 be two of these, separated by Δt and $\Delta\mathbf{v} = (0, \beta_y'' c, 0)$

$\Delta\boldsymbol{\Omega} = -(\gamma-1)(\mathbf{v}\times\Delta\mathbf{v})/v^2$ which leads to the

Thomas precession frequency

$$\boxed{\boldsymbol{\omega} = d\boldsymbol{\Omega}/dt = -(\gamma-1)(\mathbf{v}\times\mathbf{a})/v^2}$$

SPACETIME

Four-vector notation

A four vector A is a vector with one time like component (index 0), and three spacelike components (index 1→3)

$$x^\mu = (x^0, x^1, x^2, x^3)$$

By convention, we choose Greek index letters (such as α, β, μ) to represent indexes $0 \rightarrow 3$, and Latin index letters (such as i) to represent indexes $1 \rightarrow 3$.

The components of a four-vector can be expressed in a given **coordinate basis** $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, such that a point in spacetime is given as $x^\mu \mathbf{e}_\mu$

Scalar product and Metric Tensor

The metric tensor we use in spacetime is

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The scalar product between two Four-vectors is

$$u \cdot v = u^\alpha b^\beta g_{\alpha\beta} = u^0 v^0 - u^1 v^1 - u^2 v^2 - u^3 v^3$$

The four-vector is sometimes referred to as the **contravariant**, and the 1-form referred to as the **covariant**. Note that $v_\alpha u^\alpha$ always **Lorentz invariant**. Meaning no change in Lorentz transform.

1-Form

The 1-form of a four-vector u is

$$u_\alpha = g_{\alpha\beta} u^\beta = (u^0, -u^1, -u^2, -u^3)$$

We can write the dot product now as

$$v \cdot u = v_\alpha u^\alpha$$

Electromagnetism

Given the three-momentum $\mathbf{p}$ of a particle, with charge q, velocity $\mathbf{v}$, then

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields

Curves and the Tangent Vector

Given an arbitrary one-dimensional curve $\mathcal{P}$ in spacetime, where the curve is described by the parameter λ, such that for a given λ, a point on the curve can be written as

$$x^0(\lambda),\ x^1(\lambda),\ x^2(\lambda), x^3(\lambda)$$

At the start of the curve, we have event $\mathcal{A}$, and at the end of the curve we have event $\mathcal{B}$. Since λ represents how far along the curve we are, the four-vector from $\mathcal{A}$ to $\mathcal{B}$ can be given by the tangent vector:

$$v = \left(\frac{d\mathcal{P}}{d\lambda}\right)_{\lambda=0}$$

We assume λ to be continuous, there is therefor set of possible tangent vectors, and the set of those is called the **vector field**. The set of possible magnitudes of the tangent vector is called the **scalar field**.

Time-like curves and Four-Velocity

The parameter for the curve is often chosen to be the proper time τ, and the laboratory coordinates becomes:

$$x^0(\tau) = ct(\tau),\quad x^1(\tau) = x(\tau),\quad x^2(\tau) = y(\tau),\quad x^3(\tau) = z(\tau)$$

Since we now use the proper time as parameter, we denote the tangent vector as the four-velocity u of the particle traveling along $\mathcal{P}$

$$u^0 = \frac{dx^0}{d\tau} = \gamma c \qquad u^i = \gamma \frac{dx^i}{dt} = \gamma v^i$$

RELATIVISTIC COLLISIONS #1

Center-of-Momentum (COM) frame

Frame where total momentum of all particles is zero.

Four-momentum (a conserved quantity)

$$p^\mu = (p^0, p^1, p^2, p^3) = (E/c. p_x, p_y p_z)$$

Particle collisions | Part 1

Two particles of mass m_1 and m_2 collide and produces a set of particles with mass m_r, $r = 3,4,5, \ldots$

The total four-momentum in the COM frame (primed) $P^{\mu\prime}$ is equal to

$$p_1^{\mu\prime} + p_2^{\mu\prime} = P^{\mu\prime} = (E'/c, 0,0,0)$$

It is convenient to look at the COM system as a system of a composite mass particle, and by the **energy-momentum relation**

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

we can write $M = E'/c^2$ since $p' = 0$, which yields

$$P^{\mu\prime} = (Mc, 0,0,0)$$

The quantity $P_\mu{}' P^{\mu\prime}$ is Lorentz invariant, meaning

$$P_\mu P^\mu = P_\mu{}' P^{\mu\prime} = M^2 c^2$$

where unprimed is lab frame. It can also be given as

$$P_\mu P^\mu = (m_1^2 + m_2^2)c^2 - 2p_{1\mu} p_2^\mu$$

by the initial particles in the lab frame.

We can thus write

$$E'^2 = M^2 c^4 = (m_1 + m_2)^2 c^4 + 2(E_1 E_2 - c^2 \mathbf{p}_1 \mathbf{p}_2)$$

Suppose particle 2 is initially stationary in the *lab. frame*, then $\mathbf{p}_2 = 0$ and $E_2 = m_2 c^2$ and we have

$$E'^2 = M^2 c^4 = (m_1^2 + m_2^2)\, c^4 + 2E_1 m_2 c^2$$

Which can be simplified to

$$E'^2 = M^2 c^4 = (m_1 + m_2)^2 c^4 + 2 m_2 c^2 T_1$$

where T_1 is particle 1 kinetic energy in lab frame. Note that E' increases as the root of T_1.

Lowest energy threshold for a reaction (other then elastic scattering) happens when all product particles have zero momentum. Total four-momentum after reaction is noted as $P^{\mu\prime\prime}$. At threshold we have

$$P_\mu'' P^{\mu\prime\prime} = (\Sigma_r m_r)^2 c^2$$

which by conservation must equal $P_\mu P^\mu$, this gives us a **threshold energy** of

$$\boxed{\frac{T_1}{m_1 c^2} = \frac{Q^2 + 2Q(m_1 + m_2)c^2}{2 m_1 m_2 c^4}}$$

where Q is the "**Q value of the reaction**":

$$\boxed{Q \equiv [\Sigma_r m_r - (m_1 + m_2)]c^2}$$

RELATIVISTIC COLLISIONS #2

Particle collisions | Part 2

If we are dealing with an **elastic scattering** instead $1+2 \to 3+4$. Where 3 is the scattered incident particle (1), and 4 is the recoiled target particle (2) then we have the following:

We let particle 1 initially travel along the $+z$-axis. The incident and scattered momentum vectors define a <u>plane that is invariant</u> under Lorentz transformation. We take it to be the xz-plane with no loss of generality

---

The <u>Lorentz transform</u> from COM to lab is defined by

$$\gamma = \frac{T_1 + (m_1+m_2)c^2}{\sqrt{2m_2c^2T_1 + (m_1+m_2)^2c^4}} \qquad \boldsymbol{\beta} = \frac{\mathbf{p}_1 c}{T_1 + (m_1+m_2)c^2}$$

We get by Lorentz transformation that

$$p_1^{0\prime} = \gamma\left(\frac{E_1}{c} - \beta p_1\right) \qquad p_1^{3\prime} = \gamma\left(p_1 - \frac{\beta E_1}{c}\right)$$

If θ is the angle between $\mathbf{p}_3'$ and $\mathbf{p}_1'$, then in <u>COM frame</u>

$$p_3^{1\prime} = p_1'\sin\theta \qquad p_3^{3\prime} = p_1'\cos\theta \qquad p_3^{0\prime} = p_1^{0\prime} = E_1'/c$$

which in the <u>lab frame</u> is

$$p_3^1 = p_3^{1\prime} = p_1'\sin\theta$$

$$p_3^3 = \gamma\left(p_3^{3\prime} - \beta p_3^{0\prime}\right) = \gamma\left(p_1'\cos\theta + \frac{\beta E_1'}{c}\right)$$

$$p_3^0 = \gamma\left(p_3^{0\prime} + \beta p_3^{3\prime}\right) = \gamma\left(\frac{E_1'}{c} + \beta p_1'\cos\theta\right)$$

And

$$E_3 = E_1 - \gamma^2\beta(1-\cos\theta)(p_1c - \beta E_1)$$

θ in the lab frame is noted as ϕ, and we have

$$\tan\phi = \frac{\sin\theta}{\gamma(\cos\theta + \beta c/v_1')}$$

or it can be written as

$$\tan\phi = \frac{\sin\theta}{\gamma[\cos\theta + \rho g(\rho,\epsilon_1)]}$$

where $\rho = m_1/m_2$ and $\epsilon_1 = T_1/(m_1c^2)$ and

$$g(\rho,\epsilon_1) = \frac{1+\epsilon_1+\rho}{\sqrt{2\rho\epsilon_1 + (1+\rho)^2}}$$

HAMILTONIAN FORMULATION

Lagrangian → Hamiltonian

Used to convert functions of one quantity, into functions of the conjugate quantity.

From Lagrangian to Hamiltonian:
We have $\mathcal{L} = \mathcal{L}(q,\dot q,t)$ and we want a function $\mathcal{H} = \mathcal{H}(q,p,t)$.

The differential of $\mathcal{L}$ is

$$d\mathcal{L} = \frac{\partial\mathcal{L}}{\partial q_i}dq_i + \frac{\partial\mathcal{L}}{\partial\dot q_i}d\dot q_i + \frac{\partial\mathcal{L}}{\partial t}dt$$

We can write this as

$$d\mathcal{L} = \dot p_i dq_i + p_i d\dot q_i + \frac{\partial\mathcal{L}}{\partial t}dt$$

We want a function that depends on p instead of $\dot q$, thus the Hamiltonian is generated by the **Legendre transformation**

$$\mathcal{H}(q,p,t) = \dot q_i p_i - \mathcal{L}(q,\dot q,t)$$

which has differential

$$d\mathcal{H} = \dot q_i dp_i - \dot p_i dq_i - \frac{\partial\mathcal{L}}{\partial t}dt$$

Which can also be expressed as

$$d\mathcal{H} = \frac{\partial\mathcal{H}}{\partial p_i}dp_i + \frac{\partial\mathcal{H}}{\partial q_i}dq_i + \frac{\partial\mathcal{H}}{\partial t}dt$$

Implying the $2n+1$ relations

$$q_i = \frac{\partial\mathcal{H}}{\partial p_i} \qquad -\dot p_i = \frac{\partial\mathcal{H}}{\partial q_i} \qquad \text{and} \qquad -\frac{\partial\mathcal{L}}{\partial t} = \frac{\partial\mathcal{H}}{\partial t}$$

The *first two* of which are known as the **canonical equations of Hamilton**

Hamiltonian equal to total energy

If the equations defining the generalized coordinates are time independent, then

$$\mathcal{L}_2\dot q_k\dot q_m = T$$

If the forces are derivable from a conservative potential V, then

$$\mathcal{L}_0 = -V$$

If both these are fulfilled, then $\mathcal{H} = T + V = E$

Difference from Lagrangian formalism

<u>Lagrangian formulation:</u>

A system of n degrees of freedom have n *second order* equations of motions expressed by the n degrees of freedom (generalized coordinates) q_i:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q_i}\right) - \frac{\partial\mathcal{L}}{\partial q_i} = 0$$

These require $2n$ initial values to be completely solved. The n-degrees of freedom q_i span out a n-dim. **configuration space**.

<u>Hamiltonian formulation:</u>

A system of n degrees of freedom have $2n$ *first order* equations of motions expressed in $2n$ *independent variables:*

$$q_i = \frac{\partial\mathcal{H}}{\partial p_i} \qquad \dot p_i = -\frac{\partial\mathcal{H}}{\partial q_i}$$

The $2n$ independent variables span out a $2n$-dim. **phase space**. Coords. in phase space are **canonical.**

Homogenous function of k-th degree

A function $f(x_1,\dots,x_n)$ is homogeneous of k-th degree if $f(sx_1,\dots,sx_n) = s^k f(x_1,\dots,x_n)$, where k is an integer.

Finding the Hamiltonian as a function of

In many problems, the Lagrangian can be expressed as a sum of functions *homogenous* of the generalized velocities of degree 0,1, and 2. In those cases we can write

$$\mathcal{H} = \dot q_i p_i - \mathcal{L}$$

$$\mathcal{H} = \dot q_i p_i - [\mathcal{L}_0(q_i,t) + \mathcal{L}_1(q_i,t)\dot q_k + \mathcal{L}_2(q_i,t)\dot q_k\dot q_m]$$

$\mathcal{L}_0$ is the part of $\mathcal{L}$ not dependent $\dot q_i$.

$\mathcal{L}_1$ is the coefficient of the part of $\mathcal{L}$ that is homogenous in $\dot q_i$ of first degree.

$\mathcal{L}_2$ is the coefficient of the part of $\mathcal{L}$ that is homogenous in $\dot q_i$ of second degree.

If t is not explicitly in $\mathcal{L}$, then t is not present in $\mathcal{H}$, and thus $\mathcal{H}$ is <u>constant in time</u>. If $\mathcal{H} = E$ then **energy is conserved**.

HAMILTONIAN

RELATIVISTIC MECHANICS

Time as a canonical coordinate

Time t must be treated as a canonical coordinate with a conjugate momentum.

The trajectory of a system in phase space can be marked by some parameter θ and t.

The Lagrangian Λ *in the configuration space* is

$$\Lambda(q,q',t,t') = t'\mathcal{L}\left(q,\frac{q'}{t'},t\right)$$

where primed is derivative with respect to θ.

The conjugate momentum of t is

$$p_t = \frac{\partial \Lambda}{\partial t'} = \mathcal{L} + t'\frac{\partial \mathcal{L}}{\partial t'}$$

By utilizing that $\dot{q} = q'/t'$ this can be written as

$$p_t = \mathcal{L} - \frac{q_i'}{t'}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \mathcal{L} - \dot{q}_i\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\mathcal{H}$$

Covariant Lagrangian and Hamiltonian

For a single **free particle**, we have Lagrangian

$$\Lambda(x^\mu, u^\mu) = \frac{1}{2}mu_\mu u^\mu$$

and Hamiltonian is

$$H_c = \frac{p_\mu p^\mu}{2m}$$

Where u^μ is the four-velocity of the particle, p^μ is the four-momentum of the particle, and m the particles mass.

Equations of motion

A system of one particle leads us via the covariant Hamiltonian H_c to these eight first-order equations of motions

$$\frac{dx^\alpha}{d\tau} = \frac{\partial H_c' g^{\alpha\beta}}{\partial p^\beta}, \quad \frac{dp^\alpha}{d\tau} = \frac{\partial H_c g^{\alpha\beta}}{\partial x^\beta}$$

Note that only the spatial equations (indexes 1 through 3) are of interest

CANONICAL TRANSFORMATIONS #1

All q_i cyclic

If all q_i are cyclic, then all $p_i = \alpha_i$ are constant. If additionally the Hamiltonian is constant in time, then $\mathcal{H} = \mathcal{H}(\alpha_i)$ and the equations of motions are

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial \alpha_i} = \omega_i = \text{const.} \Rightarrow q_i = \omega_i t + \beta_i$$

The Four Basic Canonical Transformations

$F = F_1(q,Q,t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$
If $F_1 = q_iQ_i$ then $Q_i = p_i$ and $P_i = -q_i$	
$F = F_2(q,P,t) - Q_iP_i$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$
If $F_2 = q_iP_i$ then $Q_i = q_i$ and $P_i = p_i$	
$F = F_3(p,Q,t) + q_ip_i$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$
If $F_3 = p_iQ_i$ then $Q_i = -q_i$ and $P_i = -p_i$	
$F = F_4(p,P,t) - Q_iP_i$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = -\frac{\partial F_4}{\partial P_i}$
If $F_4 = p_iP_i$ then $Q_i = p_i$ and $P_i = -q_i$	

Canonical transformation Harmonic Oscillator

Hamiltonian is $\mathcal{H} = \frac{1}{2m}(p^2 m\omega^2 q^2)$. If one transforms $p = f(P)\cos Q$ and $q = \sin Q\ f(P)/m\omega$ then we get $K = f^2(P)/2m$ which is cyclic for Q.

By using the first basic transform $F_1 = \cot Q\ m\omega q^2/2$ we eventually end up with $K = \omega P$ and we get

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \Rightarrow Q = \omega t + \alpha$$

Point transformations

Going from one set of generalized coordinates q_i to another.

Point transformation of configuration space:

$$Q_i = Q_i(q,t)$$

Point transformation of phase space:

$$Q_i = Q_i(q,p,t)$$
$$P_i = P_i(q,p,t)$$

When dealing with the Lagrangian, the first one is enough. But when dealing with the Hamiltonian, the generalized momenta is independent variables which also need to be considered.

Canonical transformation

Transformation from the set of coordinates (q,p) to the set (Q,P) need to satisfy

$$p_iq_i - \mathcal{H} = P_iQ_i - K + \frac{dF}{dt}$$

Where K is the Hamiltonian in the new set of coordinates (sometimes called the **Kamiltonian**), and F is a function of the phase space coordinates.

Example of canonical transformation

Suppose F where given as $F = F_1(q,Q,t)$, then

$$p_iq_i - \mathcal{H} = P_iQ_i - K + \frac{dF_1}{dt}$$
$$= P_iQ_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i$$

For both sides to be equal, we need

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i} \qquad K = H + \frac{\partial F_1}{\partial t}$$

These are the **transformation equations**

We would first solve all p_i which would become a functions of q_j, Q_j and t. Which could assumingly be inverted to become functions of Q_i, which would then be used on the middle equation to solve for all P_i. Finally, the third equation gives us the Kamiltonian, which we could then express with Q and P

CANONICAL TRANSFORMATIONS #2

Restricted canonical transformation

Time independent canonical transformation

$$Q_i = Q_i(q,p) \qquad P_i = P_i(q,p)$$

<u>Important:</u> The Hamiltonian *does not change* in such transformations $\mathcal{H} = K$

Symplectic notation for Restricted Canonical

For restricted canonical transformation, the <u>equations of motion</u> of the *old coordinates* is

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial \mathcal{H}}{\partial \boldsymbol{\eta}}$$

Where $\boldsymbol{\eta}$ is a column matrix of the $2n$ old coordinates q_i and p_i, and the $\mathbf{J}$ matrix is defined as

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}$$

i.e., a $2n \times 2n$ matrix composed by four $n \times n$ zero $\mathbf{0}$ and unit $\mathbf{1}$ matrices.

The restricted canonical transformation can be written as

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}(\boldsymbol{\eta})$$

where $\boldsymbol{\zeta}$ is a column matrix of the $2n$ new coordinates Q_i and P_i

The <u>equations of motion</u> of the *new coordinates* is

$$\dot{\zeta} = \frac{\partial \zeta_i}{\partial \eta_i}\dot{\eta}_i$$

which in symplectic notation can be written as

$$\dot{\boldsymbol{\zeta}} = \mathbf{M}\dot{\boldsymbol{\eta}} = \mathbf{MJ}\frac{\partial \boldsymbol{\mathcal{H}}}{\partial \boldsymbol{\eta}} = \mathbf{MJM}^{\intercal}\frac{\partial \mathcal{H}}{\partial \boldsymbol{\zeta}} = \mathbf{J}\frac{\partial \mathcal{H}}{\partial \boldsymbol{\zeta}}$$

where $\mathbf{M}$ is the **Jacobian matrix** of the transformation with elements

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j}$$

Symplectic vs Regular

One can use both concepts/notations. They have a connection, but that is irrelevant. Both are great tools of looking at canonical transformations.

Poisson bracket properties

$$[u,u] = 0 \qquad [u,v] = -[v,u]$$

$$[au + bv, w] = a[u,w] + b[v,w]$$

$$[uv, w] = [u,w]v + u[v,w]$$

$$\big[u,[v,w]\big] + \big[v,[w,u]\big] + \big[w,[u,v]\big] = 0$$

The last one being called **Jacobi's identity.**

Poisson bracket formulation

Time derivative of a function $u(q,p,t)$ is

$$\frac{du}{dt} = [u,\mathcal{H}] + \frac{\partial u}{\partial t}$$

If u is one of the canonical variables q_i or p_i

$$q_i = [q_i,\mathcal{H}] \qquad p_i = [p_i,\mathcal{H}]$$

$$\dot{\boldsymbol{\eta}} = [\boldsymbol{\eta},\mathcal{H}]$$

If u is $\mathcal{H}$ itself then

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}$$

If u is constant of motion, then

$$[\mathcal{H},u] = \frac{\partial u}{\partial t}$$

Infinitesimal canonical transformation (I.C.T)

$$Q_i = q_i + \delta q_i \qquad P_i = p_i + \delta p_i$$

In matrix form: $\boldsymbol{\zeta} = \boldsymbol{\eta} + \delta\boldsymbol{\eta}$

A suitable generation function would be

$$F_2 = q_i P_i + \epsilon G(q,P,t)$$

wherein we can write

$$\delta\boldsymbol{\eta} = \epsilon\mathbf{J}\frac{\partial G}{\partial \boldsymbol{\eta}} = \epsilon[\boldsymbol{\eta}, G]$$

If $G = \mathcal{H}$, and we let $\epsilon = dt$ then

$$\delta\boldsymbol{\eta} = dt[\boldsymbol{\eta},\mathcal{H}] = \dot{\boldsymbol{\eta}}dt = d\mathbf{n}$$

i.e., the evolution of a system is a continuous application of I.C.T's with the Hamiltonian as the generator function.

Let u be a function of the system config. and each point in phase-space is given by a parameter α:

$$u(\alpha) = u_0 + \alpha[u,G]_0 + \frac{\alpha^2}{2!}\big[[u,G],G\big]_0 + \frac{\alpha^3}{3!}\Big[\big[[u,G],G\big],G\Big]_0 + \cdots$$

Poisson bracket

Poisson bracket of two functions u and v with respect to canonical variables (q,p) is

$$[u,v]_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

In matrix form

$$[u,v]_{\boldsymbol{\eta}} = \left(\frac{\partial u}{\partial \boldsymbol{\eta}}\right)^{\intercal}\mathbf{J}\left(\frac{\partial v}{\partial \boldsymbol{\eta}}\right)$$

If u and v are canonical coordinates themselves

$$\big[q_j, q_k\big]_{q,p} = 0 = \big[p_j, p_k\big]_{q,p}$$

$$\big[q_j, p_k\big]_{q,p} = \delta_{jk} = -\big[p_j, q_k\big]_{q,p}$$

In matrix form

$$[\boldsymbol{\eta},\boldsymbol{\eta}]_{\boldsymbol{\eta}} = \mathbf{J}$$

where the square matrix Poisson bracket $[\boldsymbol{\eta},\boldsymbol{\eta}]$ has elements lm equal to $[\eta_l,\eta_m]$

All Poisson brackets are invariant under canonical transformation

Invariant phase space volume

After a canonical transform $\boldsymbol{\eta} \to \boldsymbol{\zeta}$ the phase space volume element is conserved

$$(d\eta) = dq_1 \dots dq_n dp_1 \dots dp_n$$
$$= (d\zeta) = dQ_1 \dots dQ_n dQ_1 \dots dQ_n$$

and the phase space volume is invariant

I.C.T rotation and Canonical Angular Momentum

Imagine a I.C.T where we rotate the system by $d\theta$ around the z-axis

$$\delta x_i = -y_i d\theta \qquad \delta y_i = x_i d\theta \qquad \delta z_i = 0$$

$$\delta p_{ix} = -p_{iy}d\theta \qquad \delta p_{iy} = p_{ix}d\theta \qquad \delta p_{iz} = 0$$

This corresponds to $G = x_i p_{iy} - y_i p_{ix}$ and $\epsilon = d\theta$

The generating function G is the z-component of the total **canonical angular momentum**

$$G = L_z \equiv (r_i \times p_i)_z$$

More generally, given a unit vector $\mathbf{n}$ for the axis rotated about, we have $G = \mathbf{L}\cdot\mathbf{n}$

CANONICAL TRANSFORMATIONS #3

Three-dimension Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} -1 & \text{if } (i,j,k) \text{ is } (z,y,x),(x,z,y), \text{or } (y,x,z) \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i \\ 1 & \text{if } (i,j,k) \text{ is } (x,y,z),(y,z,x), \text{or } (z,x,y) \end{cases}$$

Rotation of system vectors

If $\mathbf{F}$ is a vector function of the only the system config. (q,p) (i.e. a system vector), then the change in $\mathbf{F}$ by an I.C.T rotation $d\theta$ about an axis defined by unit vector $\mathbf{n}$ is

$$d\mathbf{F} = d\theta[\mathbf{F}, \mathbf{L}\ \cdot \mathbf{n}] = \mathbf{n} d\theta \times \mathbf{F}$$

which implies the Poisson bracket identities

$$[\mathbf{F}, \mathbf{L}\ \cdot \mathbf{n}] = \mathbf{n} \times \mathbf{F}$$
$$[\mathbf{F}\cdot\mathbf{G}, \mathbf{L}\cdot\mathbf{n}] = 0$$
$$[L^2, \mathbf{L}\cdot\mathbf{n}] = 0$$
$$[L_i, L_j] = \epsilon_{ijk} L_k$$
$$[p_i, L_j] = \epsilon_{ijk} p_k$$

One common example: If $\mathbf{F} = \mathbf{p}$ and $\mathbf{n} = \mathbf{k}$ then

$$[p_x, xp_y - yp_x] = -p_y$$
$$[p_y, xp_y - yp_x] = p_x$$
$$[p_z, xp_y - yp_x] = 0$$

Solving mechanical problems with Canonical transformations | TWO METHODS

1. If $\mathcal{H}$ is conserved, then the equations of emotions are trivial to find if one does a canonical transformation to new canonical coordinates that lead to all q_i being cyclic
2. Find a canonical transformation from (q,p) at time t to (q_0,p_0) at time t_0. The equations of transformations
$$q = q(q_0, p_0, t)$$
$$p = p(q_0, p_0, t)$$
will be the equations of motions.

HAMILTON-JACOBI THEORY

Hamilton-Jacobi equation and method

$$\mathcal{H}\left(q_1, \dots, q_n; \frac{\partial F_2}{\partial q_i}, \dots, \frac{\partial F_2}{\partial q_n};\ t\right) + \frac{\partial F_2}{\partial t} = 0$$

The generating function

$$F_2 = S = S(q_1, \dots, q_n;\ \alpha_1, \dots, \alpha_n;\ t)$$

which fulfills this equation is called **Hamilton's principal function** and assures that the new coordinates Q, P are constant in time

---

We end up with the **transformation equations**

$$P_i = \alpha_i = \text{"const."} \quad p_i = \frac{\partial S}{\partial q_i} \quad Q_i = \frac{\partial S}{\partial \alpha_i} = \beta_i = \text{"const."}$$

Which can be used to find

$$q_j = q_j(\alpha, \beta, y) \quad \text{and} \quad p_j = p_j(\alpha, \beta, t)$$

Which is the Hamiltonian equations of motion where α and β is found by given some initial value q_0, p_0 at $t = t_0$

If $\mathcal{H}$ not explicitly dependent on time, t, then

$$S(q, \alpha, t) = W(q, \alpha) - at$$

Hamilton-Jacobi method 1

If $\mathcal{H}$ is conserved, then we have the **restricted Hamilton-Jacobi equation**

$$\mathcal{H}\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_i$$

Since $\mathcal{H}$ not expl. dep. on time, we have $\mathcal{H} = K = \alpha_1$ thus W generates canonical transformations where all new coordinates Q_i are cyclic. We thus have $P_i = \alpha_i$ and

$$Q_1 = t + \beta_1 = \frac{\partial W}{\partial \alpha_1} \qquad Q_i = \beta_i = \frac{\partial W}{\partial \alpha_i}, i \neq 1$$

Separation of variables

If $S(q,\alpha,t) = S_1(q_1,\alpha,t) + S'(q_{i\neq 1},\alpha,t)$ then we can separate Hamil-Jaco. into one equation for q_1 and $n-1$ for $q_{i\neq 1}$. It is completely separable if $S = \Sigma_i s_i$ wherein we can have n Hamil-Jaco. equations.

Hamilton-Jacobi method 2 example

One dimensional harmonic oscillator

$$\mathcal{H} = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) = E \qquad \omega = \sqrt{k/m}$$

We set $p = \partial S/\partial q$ and write $\mathcal{H}$ as a function of q and $\partial S/\partial q$ and thus have the Hamilton-Jacobi

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial S}{\partial t} = 0$$

$\mathcal{H}$ not expl. dep. on t, thus $S = W(q,\alpha) - at$

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha \quad \Rightarrow \quad E = \alpha$$

By immediate integration we obtain

$$W = \alpha \int \sqrt{1 - m\omega^2 q^2/(2\alpha)}\, dq$$
$$S = \sqrt{2m\alpha} \int \sqrt{1 - m\omega^2 q^2/(2\alpha)}\, dq - \alpha t$$

$$\beta = \frac{\partial S}{\partial \alpha} = \operatorname{asin}(q)\sqrt{\frac{m\omega^2}{2\alpha}} - t$$

We thus have the equations of motions

$$\Rightarrow q = \sqrt{\frac{2\alpha}{m\omega^2}} \sin(\omega t + \beta)$$

$$p = \frac{\partial S}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2} = \sqrt{2m\alpha}\cos(\omega t + \beta)$$

Two dimensional anisotropic harmonic oscillator

$$\mathcal{H} = \frac{1}{2m}\left(p_x^2 + p_y^2 + m^2\omega_x^2 x^2 +, m^2\omega_y^2 y^2\right) = E$$

$$\omega_x = \sqrt{k_x/m} \qquad \omega_y = \sqrt{k_y/m}$$

Here the coordinates and momenta separate into two to sets, we can thus write

$$S(x, y, \alpha, \alpha_y, t) = W_x(x,\alpha) + W_y(y, \alpha_y) - \alpha t$$

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + m^2\omega_x^2 x^2\right] = \alpha_x$$

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial y}\right)^2 + m^2\omega_y^2 y^2\right] = \alpha_x$$

with $\alpha = \alpha_x + \alpha_y = E$

CONTINOUS SYSTEMS

Continuous index

Just as i is used to represent different discrete indexes of generalized coordinates η_i, we use continuous indexes x, y, z (one for each dimension) to represent continuous coordinates $\eta(x,y,z,t)$.

Lagrangian density

For continuous systems, we have

$$\mathcal{L} = \int\int\int \mathcal{D}\, dxdydz$$

where $\mathcal{D}$ is the Lagrangian density

Elastic rod example

Can be first thought of as a discrete system of mass points with mass m separated by springs with stiffness k, where the displacement of each mass point is η_i, where a is the displacement distance between the points. Using Hooke's law, we get

$$\mathcal{L} = \frac{1}{2}\sum_i a\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1}-\eta_i}{a}\right)^2\right] = \sum_i a\mathcal{L}_i$$

In a continuous elastic rod, we instead have instead a continuous **field variable/quantity** $\eta(x,t)$

$\left(\frac{\eta_{i+1}-\eta_i}{a}\right) \to \left(\frac{\eta(x+a,t)-\eta(x,t)}{a}\right)$ and $a \to dx$,

thus $\frac{m}{a} \to \mu$ is the mass per length, $ka \to Y$ is the <u>Youngs Modulus</u>. The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\int dx\left[\mu\left(\frac{d\eta}{dt}\right)^2 - Y\left(\frac{d\eta}{dx}\right)^2\right]$$

Lagrange-Euler equations (field equations)

$$\frac{d}{dx^\beta}\left(\frac{d\mathcal{D}}{\partial\eta_{\alpha,\beta}}\right) - \frac{\partial\mathcal{D}}{\partial\eta_\alpha} = 0$$

where $x^\beta = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and

$$\eta_{\alpha,\beta} = \frac{d\eta_\alpha}{dx^\beta}; \quad \eta_{,j} = \frac{d\eta}{dx^j}; \quad \eta_{i,\alpha\beta} = \frac{d^2\eta_i}{dx^\alpha dx^\beta}$$

where Greek letters refer to indices 1 to 3 and Lating letters refer to indices 0 to 3.

The Lagrangian is denoted as $\mathcal{L}\int\mathcal{D}(dx^i)$

Stress-Energy tensor forms in component notation

$T^{\alpha\beta} = T^{\beta\alpha}$ is the *contravariant form*

$T_{\mu\nu} = T^{\alpha\beta} g_{\alpha\mu} g_{\beta\nu}$ is the *covariant form*

${T^\mu}_\nu = T^{\alpha\beta} g_{\alpha\nu}$ is the *mixed form*

Stress-Energy Tensor properties

Let u a four-velocity of a observer. Let V be 3D volume. The stress-energy tensor $\overleftrightarrow{\mathbf{T}}$ has "slots for two vectors"

If u is inserted into one of the vector slots, we get the output

$$\overleftrightarrow{\mathbf{T}}(u,\_) = \overleftrightarrow{\mathbf{T}}(\_,u) = -\left(\text{density of 4-momentum}, \frac{d\mathbf{p}}{dV}\right)$$

In component notation:

$$T^\alpha{}_\beta\, u^\alpha = T_\beta{}^\alpha\, u^\beta = -\left(\frac{dp^\alpha}{dV}\right)$$

Result is the negative of the 4-momentum per unit 3D volume measured in the observer ref. frame at the event where $\overleftrightarrow{\mathbf{T}}$ is measured.

Let n be an arbitrary four-unit vector, then if inserted in the other slot:

$$T(u,n) = T(n,u) = -\left(n\cdot\frac{d\mathbf{p}}{dV}\right)$$

In component notation:

$$T_{\alpha\beta}u^\alpha n^\beta = T_{\beta\alpha}u^\beta n^\alpha = -n_\mu\left(\frac{dp^\mu}{dV}\right)$$

Result is the negative of the component of the 4-momentum density along the n direction.

If both slots are u

$$T(u,u) = \text{"mass energy per unit volume"}$$

In component notation:

$$T_{\alpha\beta}u^\alpha u^\beta = T_{\beta\alpha}u^\beta u^\alpha = u_\mu\frac{dp^\mu}{dV}$$

Result is mass energy per unit volume as measured in frame with u.

If a frame is picked, and we insert two spacelike basis vectors e_i and e_j in that frame, the result is

$$T_{ij} = T_{ji} = \overleftrightarrow{T}(e_i, e_j) = \overleftrightarrow{T}(e_j, e_i)$$

$= i$-componment of force acting from side $x^j - \delta$ to side $x^j + \delta$ across unit surface area perpendicular to e_j

$= j$-componment of force acting from side $x^i - \delta$ to side $x^i + \delta$ across unit surface area perpendicular to e_i

Stress-Energy Tensor

Describes the density and flux of energy and momentum in spacetime.

$T^{\alpha\beta}$ gives the flux of the α-th component of the 4-momentum vector across the surface with constant x^β coordinate.

The tensor can be displayed as a 4×4 matrix

$$T^{\alpha\beta} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}$$

A definition of Stress-Energy Tensor

$$\frac{\partial\mathcal{D}}{\partial x^\mu} = -\frac{d}{dx^\nu}\left[\frac{\partial\mathcal{D}}{\partial\eta_{\rho,\nu}}\eta_{\rho,\mu} - \mathcal{D}\delta_{\mu\nu}\right]$$

If $\mathcal{D}$ not expl. dep. on x^μ then

$$\frac{d}{dx^\nu}\left[\frac{\partial\mathcal{D}}{\partial\eta_{\rho,\nu}}\eta_{\rho,\mu} - \mathcal{D}\delta_{\mu\nu}\right] = \frac{dT_\mu{}^\nu}{dx^\nu} = T_\mu{}^\nu{}_{,\nu} = 0$$

where $T_\mu{}^\nu = \frac{\partial\mathcal{D}}{\partial\eta_{\rho,\nu}}\eta_{\rho,\mu} - \mathcal{D}\delta_{\mu\nu}$

Stress-Energy Tensor with Perfect Fuid

Perfect fluid moving with four-velocity u in space-time with mass-density ρ and isotropic pressure p in the rest frame of the fluid.

Stress energy tensor is given by

$$\overleftrightarrow{T} = (\rho + p)u\otimes u + pg$$

in component form

$$T_{\alpha\beta} = (\rho+p)u_\alpha u_\beta + pg_{\alpha\beta}$$

Inserting u into one of the "slots" we get

$$T^\alpha{}_\beta\, u^\beta = \left[(\rho+p)u^\alpha u_\beta + p\delta^\alpha{}_\beta\right]u^\beta = \rho u^\alpha$$

In the rest frame we get

$$T^0{}_\beta u^\beta = \rho c$$

$$T^i{}_\beta = \frac{dp^i}{dV} = \text{momentum density} = 0$$

Thus $T_{ik} = \overleftrightarrow{T}(e_i, e_k) = p\delta_{ik}$

RELATIVISTIC FIELD THEORIES

Klein-Gordon (complex scalar field example)

The Lagrangian density will be given by 2 independent field variables ϕ and ϕ^*which are 4-scalars and conjug.

Let $\mathcal{D} = c^2\phi_{,\lambda}\phi^{*,\lambda} - \mu_0^2 c^2 \phi\phi^*$ where μ_0 const and

$$\phi_{,\lambda} = \frac{\partial \phi}{\partial x^\lambda} \qquad \phi^{*,\lambda} = g^{\lambda\nu}\frac{\partial \phi}{\partial x^\nu}$$

Expressed in terms of space and time we have

$$\mathcal{D} = \dot{\phi}\dot{\phi}^* - c^2\nabla\phi\cdot\nabla\phi^* - \mu_0^2c^2\phi\phi^*$$

This Lagrangian density is Lorentz invariant.

To obtain the field equations for $\eta_\rho = \phi^*$ we note

$$\frac{\partial \mathcal{D}}{\partial \eta_{\rho,\nu}} = \frac{\partial \mathcal{D}}{\partial \phi^*_{,\nu}} = c^2\phi^{,\nu} \qquad \frac{\partial \mathcal{D}}{\partial \eta_\rho} = \frac{\partial \mathcal{D}}{\partial \phi^*} = -\mu_0^2c^2\phi$$

which when put into the Lagrange-Euler equations give

$$\frac{d}{dx^\nu}(c^2\phi^{,\nu}) + \mu_0^2c^2\phi = 0 \quad \Rightarrow \quad \frac{d}{dx^\nu}(\phi^{,\nu}) + \mu_0^2\phi = 0$$

$$\Rightarrow \phi_{,\nu}{}^{\nu} + \mu_0^2\phi = 0$$

Which is the same as

$$\sum_\nu \frac{d^2\phi}{(dx^\nu)^2} + \mu_0^2\phi = 0$$

The equation of such satisfied by both ϕ and ϕ^* is the **Klein-Gordon equation** and is a relativistic analog of the Schrödinger Equation for a charged zero-spin particle with rest mass μ_0.

sine-Gordon (real scalar field)

From the example to the left, if the scalar field is real (i.e., $\phi = \phi^*$), and to only exist in one spatial dim, then

$$\mathcal{D} = \frac{1}{2}\left[\left(\frac{d\phi}{dt}\right)^2 - c^2\left(\frac{\partial\phi}{\partial x}\right)^2 - \mu_o^2c^2\phi^2\right]$$

Introduction of 1/2 is for convenience and does not change the equations of motion.

We get the field equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \mu_o^2\phi^2$$

which is the *one-dimensional Klein-Gordon* eq.

We can look at $\mathcal{D}$ as a approximation to

$$\frac{1}{2}\left[\left(\frac{d\phi}{dt}\right)^2 - c^2\left(\frac{\partial\phi}{\partial x}\right)^2\right] - \mu_o^2c^2\phi^2(1-\cos\phi)$$

Which has field eq.

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \mu_o^2\sin\phi$$

Which is the **sine-Gordon equation**, also known as the **pendulum equation**.

Electromagnetic four-potential

$A^\mu = \left(\frac{\phi}{c}, \mathbf{A}\right)$ where ϕ is electric potential and $\mathbf{A}$ is the magnetic potential. $\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$

Electromagnetic field

Let the components A^μ be treated as the field quantities, then

$$\mathcal{D} = -\frac{F_{\lambda\rho}F^{\lambda\rho}}{4} + j_\lambda A^\lambda$$

where F is the **Faraday-tensor**

$$cF_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

and j is the current density.

To obtain the Euler-Lagrange equations, we note

$$\frac{\partial\mathcal{D}}{\partial A^\mu} = j_\mu; \quad \frac{\partial\mathcal{D}}{\partial A_{\mu,\nu}} = -\frac{F_{\lambda\rho}}{2}\frac{\partial F^{\lambda\rho}}{\partial A_{\mu,\nu}}$$

Which gives us

$$\frac{dF^{\mu\nu}}{dx^\nu} - \sqrt{\frac{\mu_0}{\epsilon_0}}j^\mu = 0$$

NOETHERS THEOREM

Noethers theorem considers I.C.Ts on the form

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu$$

And the effects of transform, of the field quantities

$$\eta_\rho(x^\mu) \to \eta'_\rho(x'^\mu) = \eta_\rho(x^\mu) + \delta\eta_\rho(x^\mu)$$

Three assumed conditions

1. We are in flat space-time
2. Lagrangian density displayes the same *functional form* in the old and new quantities:
$$\mathcal{D}'\big(\eta'_\rho(x'^\mu), \eta'_{\rho,\nu}(x'^\mu), x'^\mu\big) = \mathcal{D}\big(\eta'_\rho(x'^\mu), \eta'_{\rho,\nu}(x'^\mu), x'^\mu\big)$$
3. Magnitude of the actins S is invariant under transformation. (**Scale invariance**)

Main conclusion of theorem

$$\frac{d}{dx^\nu}\left[\left(\frac{\partial\mathcal{D}}{\partial\eta_{\rho,\nu}}\eta_{\rho,\sigma} - \mathcal{D}\delta^\nu_\sigma\right)X^\sigma_r - \frac{\partial\mathcal{D}}{\partial\eta_{\rho,\nu}}\Psi_{r\rho}\right] = 0$$

where $\delta x^\nu = \epsilon_r X^\nu_r$ and $\delta\eta_\rho = \epsilon_r\Psi_{r\rho}$

with ϵ_r being R infinitesimal parameters $r = 1, 2, \ldots R$

<u>In words:</u> If the system (or Lagrangian density) has symmetry properties such that conditions 2 and 3 hold, then there exist r conserved quantities.

Main conclusion of theorem (simple case)

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial\dot{q}_k}\dot{q}_k - L\right)X_r - \frac{\partial L}{\partial\dot{q}_k}\Psi_{rk}\right] = 0$$

which is for discrete mechanical systems.

REFERENCES

[1] Lionel Brits "Euler angles, vector of File:Euler.png.", Wikimedia Commons, https://commons.wikimedia.org/wiki/File:Right_hand_rule_cross_product.svg (accessed May. 11, 2025). (Original work remixed by me) License CC BY-SA 3.0 Deed