Largest Rectangle Inside Any Given Right Triangle

Augustin Winther

Department of Physics and Technology, University of Bergen

Monday 09th October, 2023

Abstract — In this paper we use trigonometry and real analysis to show that the largest rectangle inside any given right triangle has an area equal to half of the area of the right triangle. We also show that the largest rectangle inside any given right triangle has either one side laying on and parallel with the hypotenuse with its remaining vertices touching half way along each of the catheti, or it has two sides laying half way along and on each of the catheti with the remaining vertex touching the hypotenuse.

I Defining any right triangle

We will define an arbitrary right triangle in a x-y Cartesian coordinate system, where the right angle vertex is at the origin and the catheti are parallel to each of the coordinate axes. Let the cathetus along the positive y-axis have length H, and let the cathetus along the positive x-axis have length W (See Fig. 1).

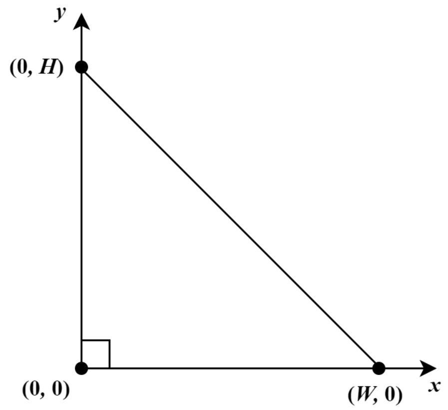

Fig. 1.

With $H > 0$ and $W > 0$, we can, by this definition, define any right triangle possible.

II A trivial observation

If we try to draw an arbitrary rectangle inside a given right triangle, such that the rectangle seems to be as large as possible, we quickly observe that no matter how much we try to rotate or stretch and squish the rectangle, it will always have at least one vertex on the hypotenuse and at least one vertex on each of the catheti (See Fig. 2 for examples).

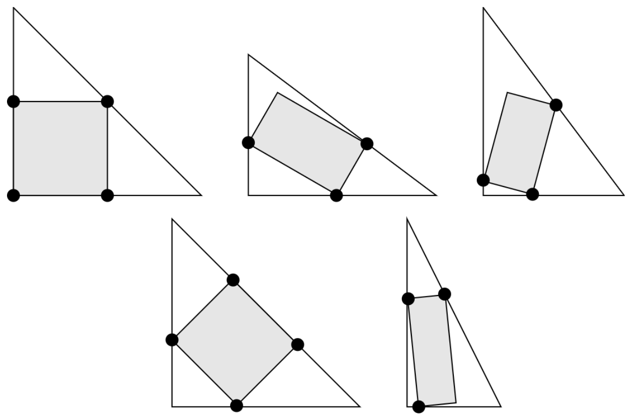

Fig. 2.

By this trivial observation, it is safe to assume that the *largest* rectangle in any given right triangle will also have this property, meaning it will have at least one vertex on the hypotenuse and at least one vertex on each of the catheti.

III Defining the rectangle

Based on our observation from Section II, we can define such a rectangle as follows:

- Let there be a variable point $A = (0, y)$ along the cathetus on the y-axis, and a variable point $B = (x, 0)$ along the cathetus on the x-axis.
- Connect points A and B by a straight line, thus creating the right triangle $\triangle AOB$, where O is the origin of the coordinate system.
- Let the angle φ equal the angle $\angle BAO$, and let the angle θ equal the top most angle in the main right triangle (See Fig. 3).

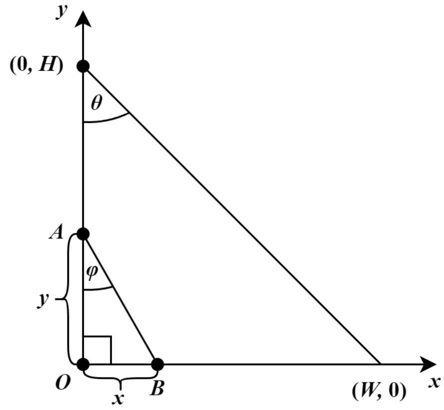

Fig. 3.

We want $0 \le \varphi \le \theta$ where $\theta < \frac{\pi}{2}$ *(More on this in Section V)*. By simple trigonometry we have that

$$\begin{aligned} 0 &\le \varphi \le \theta \\ \Rightarrow \tan(0) &\le \tan(\varphi) \le \tan(\theta) \\ \Rightarrow 0 &\le \frac{x}{y} \le \frac{W}{H} \\ \Rightarrow 0 &\le xH \le yW \\ \Rightarrow 0 &\le x \ , \ 0 \le y \end{aligned} \tag{1}$$

- Draw a line from A onto a point $C = (x_C, y_C)$ on the hypotenuse such that this new line is perpendicular to the line connecting A and B.
- Define the length of the line from A to B as h, and the length of the line from A to C as w.
- Complete the rectangle by drawing a line of length w from B and a line with length h from C such that their ends meet and we complete the rectangle (See Fig. 4).

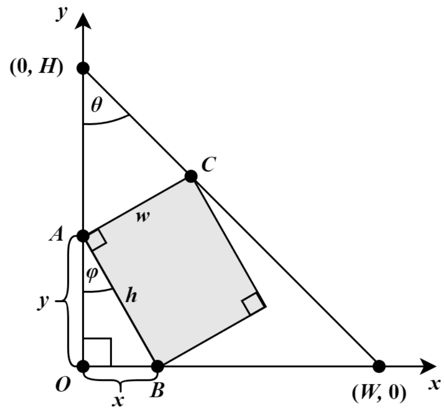

Fig. 4.

Thus, given H and W, we have now defined a semi-arbitrary rectangle in a given right triangle, where the rectangle is only dependent on the variables x and y. Since points A and B are defined to be along the catheti, and given the limitations from Eq. 1, we have that $x \in [0, W]$, $y \in [0, H]$ and $xH \in [0, yW]$.

IV Area of the rectangle

The area of the rectangle is equal to $w \cdot h$, and since the rectangle is only defined by the variables x and y, we thus have to find the functions $h(x, y)$ and $w(x, y)$.

The function $h(x, y)$ is trivially found by the Pythagorean theorem.

$$h(x, y) = \sqrt{x^2 + y^2} \tag{2}$$

To find $w(x, y)$ we need to do some more trigonometric trickery. In Section III, we defined w as the length of the line connecting A and C. We thus have that

$$w = \sqrt{(x_C - 0)^2 + (y_C - y)^2} \tag{3}$$

Let $y_C - y = k$. The placement of C is obviously dependent on the angle φ, which in turn means that k and x_C are also dependent on φ. This dependency is easy enough to see if we draw a horizontal line from the y-axis to C as shown in Fig. 5.

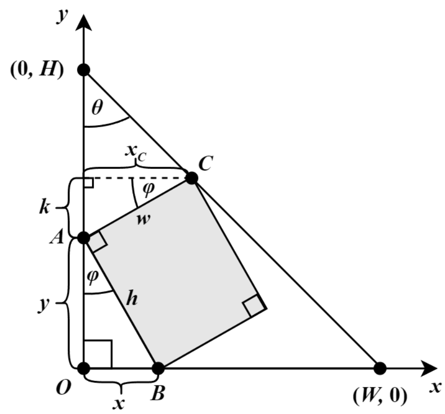

Fig. 5.

This illustration shows that

$$\begin{aligned} \tan(\varphi) &= \frac{k}{x_C} \\ \Rightarrow x_C &= \frac{k}{\tan(\varphi)} = \frac{k}{\frac{x}{y}} \\ \Rightarrow x_C &= k\frac{y}{x} \end{aligned} \tag{4}$$

$$\text{and} \quad y_C = y + k \tag{5}$$

Note that if a point (a, b) is on the hypotenuse of the right triangle as we have defined, then the coordinates of that point has the following property

$$\frac{a}{W} + \frac{b}{H} = 1 \tag{6}$$

and since the point C is on the hypotenuse, we have

$$\begin{aligned}
&\frac{x_C}{W} + \frac{y_C}{H} = 1 \\
&\overset{(4)(5)}{\Rightarrow} \frac{k\frac{y}{x}}{W} + \frac{y+k}{H} = 1 \\
&\Rightarrow k\left(\frac{y}{Wx} + \frac{1}{H}\right) = 1 - \frac{y}{H} \\
&\Rightarrow k\left(H\frac{y}{x} + W\right) = WH - Wy \\
&\Rightarrow k = \frac{WH - Wy}{W + H\frac{y}{x}} \\
&\Rightarrow k = \frac{Wx(H-y)}{Wx + Hy}
\end{aligned} \tag{7}$$

We now have enough information to find an expression for $w(x, y)$

$$\begin{aligned}
w &\overset{(3)}{=} \sqrt{(x_C - 0)^2 + (y_C - y)^2} \\
&\overset{(4)(5)}{=} \sqrt{\left(k\frac{y}{x}\right)^2 + k^2} \\
&= k\sqrt{\left(\frac{y}{x}\right)^2 + 1} \\
&= k\frac{1}{x}\sqrt{y^2 + x^2} \\
&\overset{(2)}{=} k\frac{1}{x}\ h(x, y) \\
&\Rightarrow w(x, y) = h(x, y)\frac{k}{x}
\end{aligned} \tag{8}$$

Finally, a function f for the area of the rectangle can be constructed

$$\begin{aligned}
f(x, y) &= h(x, y) \cdot w(x, y) \\
&\overset{(8)}{=} h(x, y) \cdot h(x, y)\frac{k}{x} \\
&= [h(x, y)]^2\frac{k}{x} \\
&\overset{(2)}{=} (x^2 + y^2)\frac{k}{x} \\
&\overset{(7)}{=} (x^2 + y^2)\frac{Wx(H-y)}{Wx + Hy}\frac{1}{x} \\
&= (x^2 + y^2)\frac{W(H-y)}{Wx + Hy}
\end{aligned} \tag{9}$$

Note that our function f is defined everywhere except for $(x, y) = (0, 0)$. To remedy this, we take a look at what happens when $(x, y) \to (0, 0)$ and expand the definition of f.

Let $y \to 0$ along $y = x$

$$\begin{aligned}
&\lim_{(x,y)\to(0,0)} (x^2 + y^2)\frac{W(H-y)}{Wx + Hy} \\
&= \lim_{x\to 0} (x^2 + x^2)\frac{W(H-x)}{Wx + Hx} \\
&= \lim_{x\to 0} 2x^2\frac{W(H-x)}{x(W+H)} \\
&= \lim_{x\to 0} 2x\frac{W(H-x)}{W+H} \\
&= 2 \cdot 0 \cdot \frac{W(H-0)}{W+H} \\
&= 0
\end{aligned} \tag{10}$$

Thus, if a limit for $f(x, y)$ as $(x, y) \to (0, 0)$ exists, it has to be zero. We can check this by using the Squeeze Theorem

$$\begin{aligned}
0 \leq |f(x, y)| &= \left|(x^2 + y^2)\frac{W(H-y)}{Wx + Hy}\right| \\
&\leq \left|(x^2 + y^2)\frac{WH}{Wx + Hy}\right| \\
&= \left|(WHx^2 + WHy^2)\frac{1}{Wx + Hy}\right| \\
&\leq \left|(WHx^2 + H^2xy + W^2xy + WHy^2)\frac{1}{Wx + Hy}\right| \\
&= \left|(Hx + Wy)(Wx + Hy)\frac{1}{Wx + Hy}\right| \\
&= |Hx + Wy|
\end{aligned} \tag{11}$$

We have that 0 and $Hx + Wy$ both tend towards 0 as $(x, y) \to (0, 0)$, therefore $\lim_{(x,y)\to(0,0)} f(x, y) = 0$ by the Squeeze Theorem. This coincides with the fact that if $x = 0$ and $y = 0$ the area of the rectangle would be zero, since points A and B would both be at the origin, and the "rectangle" would end up as a straight line from the origin to the hypotenuse.
We can thus expand our definition of $f(x, y)$ as such:

$$f(x, y) = \begin{cases} (x^2 + y^2)\frac{W(H-y)}{Wx+Hy} \text{ if } x \neq 0 \wedge y \neq 0 \\ 0 \text{ if } x = 0 \wedge y = 0 \end{cases} \tag{12}$$

V Limits of the definition

We take note that the function f from Eq. 12 only works for the rectangle we have defined, which is specially limited by $0 \leq \varphi \leq \theta$ where $\theta < \frac{\pi}{2}$. If $\varphi > \theta$, then it would be impossible to keep the rectangle completely inside the right triangle after drawing the line from A to C (See Fig. 6).

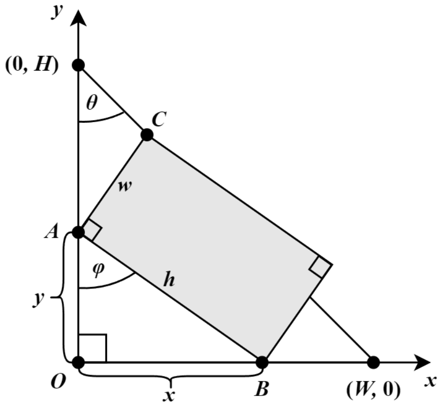

Fig. 6.

At first glance this might seem like we have limited ourselves to only be able to define half of all possible rectangles that would fit the case described in Section II, this is not the case however.
Take the following situation, illustrated in Fig. 7, where $\varphi > \theta$.

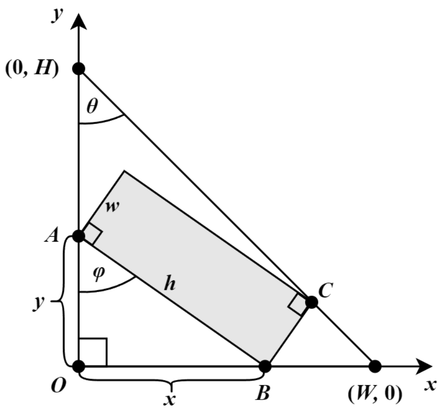

Fig. 7.

The situation illustrated in Fig. 7 will not work with our function f from Eq. 12 as C would now have to be defined by a straight line coming from B to the hypotenuse, instead of a line coming from A to the hypotenuse. A situation like this will lead to a different function for the area of the rectangle, which is left as an exercise for the reader.

However, if we flip the situation from Fig. 7 horizontally and then rotate it 90 degrees with the clock, so that the x- and y-axes switch places, we get the following situation illustrated in Fig. 8

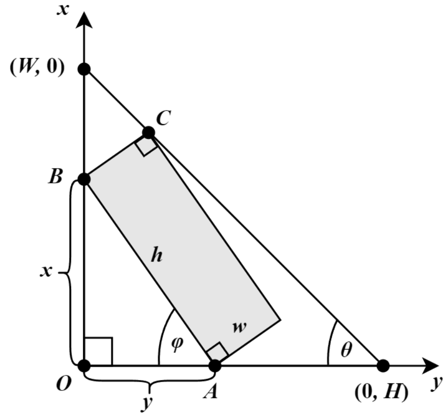

Fig. 8.

Thus, we can obviously see that any situation where $\varphi > \theta$ is equivalent to a situation where $0 \leq \varphi \leq \theta$, where the x and y values, and H and W values have switched places. Thus our definition of a rectangle from Section III can indeed define all possible rectangles that would fit the case described in Section II, including the largest possible rectangle. And further more, our function $f(x, y)$, for the area of the rectangle, will assume all possible area sizes of these rectangles.

VI The largest area

To find the largest area possible, we need to find the maximum value of f on its domain.

We already limited the domain $\mathcal{D}_f$ of our function f in Eq. 1, and it is as follows

$$\mathcal{D}_f = (x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq H \ , \ 0 \leq xH \leq yW \tag{13}$$

Since f is a continuous function (per our expansion from Eq. 12) of two variables with a bounded and closed domain $\mathcal{D}_f \subset \mathbb{R}^2$, then f will achieve a global maximum value somewhere on its domain.

Further more, we have that our function f can only achieve a global extreme value at a given point (a, b) in $\mathcal{D}_f$, if:

a) $\nabla f(a, b) = \mathbf{0}$,
b) $\nabla f(a, b)$ is undefined, or
c) (a, b) is a boundary point of $\mathcal{D}_f$

Where $\nabla f(a, b) = f_x(a, b)\mathbf{i} + f_y(a, b)\mathbf{j}$ is the gradient of f at (a, b).

Situation a)
We start of by finding the critical points of $f(x,y)$. That is, the points (a,b) where $\nabla f(a,b) = \mathbf{0}$.

$$
\begin{aligned}
&\nabla f(x,y) = \mathbf{0} \\
\Leftrightarrow\ & f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = \mathbf{0} \\
\Leftrightarrow\ & f_x(x,y) = 0 \wedge f_y(x,y) = 0
\end{aligned}
\tag{14}
$$

thus we have to solve Eq. 14 for x and y, which gives us only one critical point at

$$
x = \frac{H\left(-H + \sqrt{W^2 + H^2}\right)}{2W} \ \wedge\ y = \frac{H}{2} \tag{15}
$$

To find out whether or not this critical point is a local maximum, we do the second derivative test for functions of two variables, which generally goes as follows

$$
\begin{gathered}
P = f_{xx}(a,b),\ Q = f_{xy}(a,b),\ R = f_{yy}(a,b) \\
S = Q^2 - PR \\
S < 0 \wedge P < 0 \Rightarrow (a,b) \text{ is local max} \\
S < 0 \wedge P > 0 \Rightarrow (a,b) \text{ is local min} \\
S > 0 \Rightarrow (a,b) \text{ is saddle point} \\
S = 0 \Rightarrow \text{inconclusive}
\end{gathered}
\tag{16}
$$

When plotting the critical point x and y values from Eq. 15 into the second derivative test, we get the following

$$
\begin{gathered}
P = \frac{2W}{\sqrt{W^2 + H^2}} \\
Q = \frac{2H}{\sqrt{W^2 + H^2}} - 2 \\
R = -\frac{2W}{\sqrt{W^2 + H^2}} \\
\Rightarrow S = \left(\frac{2H}{\sqrt{W^2 + H^2}} - 2\right)^2 + \frac{4W^2}{W^2 + H^2} \\
\Rightarrow S > 0
\end{gathered}
\tag{17}
$$

Thus, the critical point from Eq. 15 is a saddle point by the second derivative test, and not a local maximum. In other words, there is no global maximum when $\nabla f(x,y) = \mathbf{0}$.

Situation b)
The next step would be to check for when the gradient does not exist

$$
\begin{gathered}
\nabla f(x,y) \text{ is undefined} \\
\Updownarrow \\
f_x(x,y) \text{ is undefined } \vee\ f_y(x,y) \text{ is undefined}
\end{gathered}
\tag{18}
$$

To determine whether Eq. 18 holds for some x and y value pair, we have to analyze the first partial derivatives f_x and f_y.

$$
\begin{aligned}
f_x &= W(H-y)\frac{-W(x^2+y^2) + 2x(Hy+Wx)}{(Hy+Wx)^2} \\
f_y &= W\frac{-H(H-y)(x^2+y^2)}{(Hy+Wx)^2} \\
&\quad + \frac{(Hy+Wx)(-x^2-y^2+2y(H-y))}{(Hy+Wx)^2}
\end{aligned}
\tag{19}
$$

Not the simplest of expressions, however, since $W > 0$ and $H > 0$, we can easily see that f_x and f_y are only undefined for $x = 0 \wedge y = 0$ (since we would get zero in the denominator of both). But since $f(0,0) = 0$, per our definition of f from Eq. 12, we would not achieve a global maximum at $(0,0)$, we would in fact achieve a global minimum. In other words, there is no global maximum when $\nabla f(x,y)$ is undefined.

Situation c)
Thus, we are left to look at the last place where $f(x,y)$ can achieve extreme values, namely the boundary of $\mathcal{D}_f$. If we look at the domain $\mathcal{D}_f$ of f from Eq. 13, we can see that we get three boundary situations:

$$
x = 0 \ \wedge\ 0 \leq y \leq H \tag{20}
$$

$$
y = H \ \wedge\ 0 \leq xH \leq yW \tag{21}
$$

$$
xH = yW \ \wedge\ 0 \leq y \leq H \tag{22}
$$

Let us start with the first boundary situation from Eq. 20 with $x = 0$ and $y \in [0, H]$

$$
\begin{aligned}
f(0,y) &= (0^2 + y^2)\frac{W(H-y)}{W \cdot 0 + Hy} \\
&= y^2\frac{W(H-y)}{Hy} \\
&= y\frac{W(H-y)}{H} \\
&= \frac{W(Hy - y^2)}{H}
\end{aligned}
\tag{23}
$$

We now find the extreme value points of $f(0,y)$

$$
\begin{aligned}
f'(0,y) &= \frac{d}{dy}\frac{W(Hy-y^2)}{H} \\
&= \frac{W(H-2y)}{H}
\end{aligned}
\tag{24}
$$

Note that $f'(0,y)$ is well defined for $y \in [0, H]$, and that for either $y = 0$ or $y = H$ we would have

$f(0, y) = 0$, which of course is not a maximum. Thus $f(0, y)$ can only achieve a maximum at a point where $f'(0, y) = 0$

$$\begin{aligned} f'(0, y) &= 0 \\ \frac{W(H - 2y)}{H} &= 0 \\ W(H - 2y) &= 0 \\ H - 2y &= 0 \\ y &= \frac{H}{2} \end{aligned} \tag{25}$$

We thus have our first potential global maximum at

$$x = 0 \quad , y = \frac{H}{2} \tag{26}$$

Moving on to the second boundary point from Eq. 21 where $y = H$ and $xH \in [0, yW]$. Looking at our function f from Eq. 12, we can instantly see that $y = H$ leads to $f(x, y) = 0$, which of course is not a maximum.

$$f(x, H) = (x^2 + H^2)\frac{W(H - H)}{Wx + H^2} = 0 \tag{27}$$

Finishing off with the third and last boundary point from Eq. 22 where $xH = yW$ and $y \in [0, H]$

$$xH = yW \ \Rightarrow \ x = y\frac{W}{H} \tag{28}$$

$$\begin{aligned} f\left(y\frac{W}{H}, y\right) &= \left[\left(y\frac{W}{H}\right)^2 + y^2\right]\frac{W(H - y)}{W\left(y\frac{W}{H}\right) + Hy} \\ &= \left(y^2\frac{W^2}{H^2} + y^2\right)\frac{W(H - y)}{y\frac{W^2}{H} + Hy} \\ &= y^2\left(\frac{W^2}{H^2} + 1\right)\frac{W(H - y)}{y\left(\frac{W^2}{H} + H\right)} \\ &= y\left(\frac{W^2}{H^2} + 1\right)\frac{W(H - y)}{\frac{W^2}{H} + H} \\ &= y\left(\frac{W^2}{H^2} + 1\right)\frac{WH(H - y)}{W^2 + H^2} \\ &= \left(\frac{W^2}{H^2} + 1\right)\frac{WH(Hy - y^2)}{W^2 + H^2} \end{aligned} \tag{29}$$

We now find the extreme value points of $f\left(y\frac{W}{H}, y\right)$

$$\begin{aligned} f'\left(y\frac{W}{H}, y\right) &= \frac{d}{dy}\left(\frac{W^2}{H^2} + 1\right)\frac{WH(Hy - y^2)}{W^2 + H^2} \\ &= \left(\frac{W^2}{H^2} + 1\right)\frac{WH(H - 2y)}{W^2 + H^2} \end{aligned} \tag{30}$$

Note that $f'\left(y\frac{W}{H}, y\right)$ is well defined for $y \in [0, H]$, and that for either $y = 0$ or $y = H$ we would have $f\left(y\frac{W}{H}, y\right) = 0$, which of course is not a maximum. Thus $f\left(y\frac{W}{H}, y\right)$ can only achieve a maximum at a point where $f'(0, y) = 0$

$$\begin{aligned} f'\left(y\frac{W}{H}, y\right) &= 0 \\ \left(\frac{W^2}{H^2} + 1\right)\frac{WH(H - 2y)}{W^2 + H^2} &= 0 \\ H - 2y &= 0 \\ y &= \frac{H}{2} \end{aligned} \tag{31}$$

Plotting $y = \frac{H}{2}$ into $x = y\frac{W}{H}$ we get $x = \frac{W}{2}$, and we thus have our last potential maximum at

$$x = \frac{W}{2} \quad , y = \frac{H}{2} \tag{32}$$

We calculate $f(x, y)$ for both potential maximum points, starting with the first one from Eq. 26

$$\begin{aligned} f\left(0, \frac{H}{2}\right) &= \left[0^2 + \left(\frac{H}{2}\right)^2\right]\frac{W\left(H - \frac{H}{2}\right)}{W \cdot 0 + H\left(\frac{H}{2}\right)} \\ &= \left(\frac{H}{2}\right)^2 \frac{W\frac{H}{2}}{\frac{H^2}{2}} \\ &= \frac{H^2}{4}\,\frac{WH}{H^2} \\ &= \frac{WH}{4} \end{aligned} \tag{33}$$

and the potential maximum point from Eq. 32

$$\begin{aligned} f\left(\frac{W}{2}, \frac{H}{2}\right) &= \left[\left(\frac{W}{2}\right)^2 + \left(\frac{H}{2}\right)^2\right]\frac{W\left(H - \frac{H}{2}\right)}{W\left(\frac{W}{2}\right) + H\left(\frac{H}{2}\right)} \\ &= \left(\frac{W^2}{4} + \frac{H^2}{4}\right)\frac{W\frac{H}{2}}{\frac{W^2}{2} + \frac{H^2}{2}} \\ &= \frac{W^2 + H^2}{4}\,\frac{WH}{W^2 + H^2} \\ &= \frac{WH}{4} \end{aligned} \tag{34}$$

As we can see, the only two candidates left for being the global maximum of f are equal, and since f has to have a global maximum, we can now conclude that the largest value f can achieve within its domain $\mathcal{D}_f$ is $\frac{WH}{4}$, and that value occurs at $\left(0, \frac{H}{2}\right)$ and $\left(\frac{W}{2}, \frac{H}{2}\right)$.

What this entails is that the maximum area a rectangle can have inside any given right triangle is $\frac{WH}{4}$ which is the same has $\frac{1}{2} \cdot \frac{WH}{2}$, or in other words, half of the area of the right triangle. If we construct a rectangle as we have defined earlier with these x and y values we get two situations. One where the rectangle has one side laying on and parallel with the hypotenuse with its remaining vertices touching half way along each of the catheti as illustrated in Fig. 9.

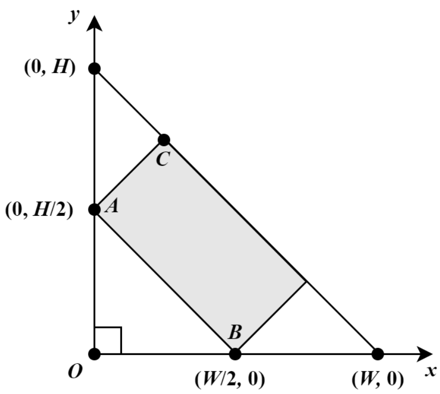

Fig. 9.

And one where the rectangle has two sides laying half way along and on each of the catheti with the remaining vertex touching the hypotenuse as illustrated in Fig. 10.

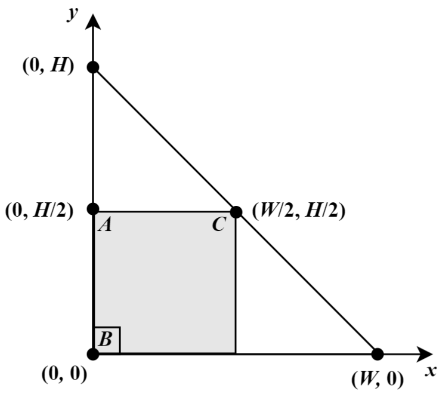

Fig. 10.

Where x_C is calculated as such

$$
\begin{aligned}
x_C &\overset{(4)}{=} k\frac{y}{x} \\
&\overset{(7)}{=} \frac{Wy(H-y)}{Wx+Hy} \qquad (35) \\
&\overset{x=0\ y=\frac{H}{2}}{=} \frac{W\frac{H}{2}\left(H-\frac{H}{2}\right)}{H\frac{H}{2}} = \frac{W\frac{H^2}{4}}{\frac{H^2}{2}} = \frac{\frac{W}{4}}{\frac{1}{2}} = \frac{W}{2}
\end{aligned}
$$

■