Mass of Both Stars in a Simple Binary Star System

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Abstract — This paper uses the center of mass equation, Newton's law of gravitation, and the formula for centripetal acceleration to derive equations for the individual masses of the two stars in a binary star system; assuming constant non-relativistic orbital speeds and circular orbits on the same plane.

Center of mass equation:

$$m_1\, r_1 = m_2\, r_2 \tag{1}$$

Given a circular orbit with constant orbital speed v, and an orbital period T, we have:

$$v = \frac{2\,\pi\, r}{T} \implies r = \frac{v\, T}{2\,\pi} \tag{2}$$

From the equations (1) and (2), we can derive the following:

$$\begin{aligned} m_1\, r_1 &= m_2\, r_2 \\ \Rightarrow \quad \frac{r_1}{r_2} &= \frac{m_2}{m_1} \\ \overset{(2)}{\Rightarrow} \quad \frac{\frac{v_1\, T}{2\,\pi}}{\frac{v_2\, T}{2\,\pi}} &= \frac{m_2}{m_1} \\ \Rightarrow \quad \frac{v_1}{v_2} &= \frac{m_2}{m_1} \end{aligned} \tag{3}$$

Let there be a constant b, representing the equal ratios from equations (1) and (3):

$$b = \frac{r_1}{r_2} = \frac{v_1}{v_2} = \frac{m_2}{m_1} \tag{4}$$

From equation (4), we note the following:

$$\begin{aligned} r_2 &= \frac{r_1}{b} \\ v_2 &= \frac{v_1}{b} \end{aligned}$$

We also note that the distance between the two masses can be written as:

$$r = r_1 + r_2 \overset{(4)}{=} r_1 + \frac{r_1}{b} = r_1 \left(1 + \frac{1}{b}\right) \tag{5}$$

Using Newton's law of gravitation and the formula for centripetal acceleration, we derive the following:

$$\begin{aligned} m_2\, a_2 &= G \frac{m_1\, m_2}{r^2} \\ \Rightarrow \quad a_2 &= G\, \frac{m_1}{r^2} \\ \Rightarrow \quad \frac{v_2^2}{r_2} &= G\, \frac{m_1}{r^2} \\ \overset{(4)}{\Rightarrow} \frac{\left(\frac{v_1}{b}\right)^2}{r_2} &= G\, \frac{m_1}{r^2} \\ \Rightarrow \quad \frac{v_1^2}{r_1 b} &= G\, \frac{m_1}{r^2} \\ \overset{(5)}{\Rightarrow} \quad \frac{v_1^2}{r_1 b} &= G\, \frac{m_1}{r_1^2\, \left(1 + \frac{1}{b}\right)^2} \\ \Rightarrow \quad m_1 &= \frac{v_1^2\, r_1^2\, \left(1 + \frac{1}{b}\right)^2}{r_1\, G\, b} \\ \Rightarrow \quad m_1 &= \frac{v_1^2\, r_1\, \left(1 + \frac{1}{b}\right)^2}{G\, b} \\ \overset{(2)}{\Rightarrow} \quad m_1 &= \frac{v_1^2\, \frac{v_1 T}{2\pi}\, \left(1 + \frac{1}{b}\right)^2}{G\, b} \\ \Rightarrow \quad m_1 &= \frac{v_1^3\, T\, \left(1 + \frac{1}{b}\right)^2}{2\pi\, G\, b} \\ \Rightarrow \quad m_1 &= \frac{\left(\frac{2\,\pi r_1}{T}\right)^3\, T\, \left(1 + \frac{1}{b}\right)^2}{2\,\pi\, G\, b} \\ \Rightarrow \quad m_1 &= \frac{8\,\pi^3\, r_1^3\, T\, \left(1 + \frac{1}{b}\right)^2}{2\,\pi\, T^3\, G\, b} \\ \Rightarrow \quad m_1 &= \frac{4\,\pi^2\, r_1^3\, \left(1 + \frac{1}{b}\right)^2}{T^2\, G\, b} \\ \Rightarrow \quad m_1 &= \frac{4\,\pi^2\, r_1^3}{T^2\, G}\, \frac{\left(1 + \frac{1}{b}\right)^2}{b} \end{aligned}$$

m_2 can be derived from m_1 and equation (4):

$$m_2 = m_1\, \frac{m_2}{m_1} \overset{(4)}{=} m_1\, b = \frac{4\,\pi^2\, r_1^3}{T^2\, G} \left(1 + \frac{1}{b}\right)^2$$